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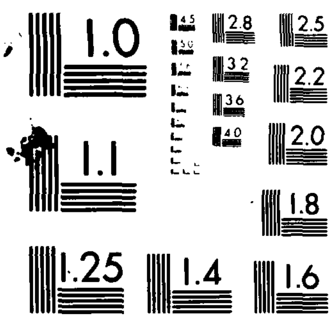
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ICE MECHANICS

Second Technical Report

by

L. W. Morland and U. Spring

December 1981

EUROPEAN RESEARCH OFFICE
United States Army
London England

CONTRACT NUMBER: DAJA37-79-C-0005

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

R&D 2595-EN

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A113 510	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Viscoelastic Solid Relations for the Deformation of Ice.		5. TYPE OF REPORT & PERIOD COVERED Second Technical Report 1 Feb - 31 Oct 81
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) U. Spring and L. W. Morland		8. CONTRACT OR GRANT NUMBER(s) DAJA37-79-C-0005
9. PERFORMING ORGANIZATION NAME AND ADDRESS School of Mathematics and Physics University of East Anglia Norwich NR4 7TJ, U.K.		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102A-IT161102 BH57-01
11. CONTROLLING OFFICE NAME AND ADDRESS USARDSG-UK Box 65 FPO New York, NY 09510		12. REPORT DATE December 1981
		13. NUMBER OF PAGES 72
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Ice mechanics, ice engineering, stress-rate, strain-rate, ice loading, ice deformation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Phase I: The finite linear viscoelastic solid and single integral representation with non-linear dependence on history are investigated in uni-axial stress. Both integral kernels in the stress formulation are determined by single-step constant strain tests, and both kernels in the strain formulation are determined by single-step constant stress tests. Single integral stress and strain formulations are not equivalent. The stress histories required to maintain constant strain-rate for both models

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are determined from the Volterra integral equations given by the strain formulations once their kernels are determined by constant stress tests. However, known constant strain-rate response does not determine the kernels Variation of the kernel within a given qualitative shape can lead to different shapes of constant strain-rate response, so that both constant stress and constant strain-rate tests may be necessary to deduce the optim single integral approximation, in preference to multi-step stress tests.

Phase II: A frame indifferent differential operator law relating stress, stress-rate, strain, and strain-rate is constructed to describe the qualitative features of both constant load and constant displacement rate response in uni-axial stress experiments. No lower order differential relation can describe both responses, but the two responses are not sufficient to determine the response coefficients of the relation. The jump relation is determined for a stress discontinuity applied in a general configuration. A simple initially isotropic model is proposed to investigate the effects of loading history and the anisotropy induced in a configuration following a loading-unloading cycle. Both uni-axial stress and shear cycles are treated, followed by uni-axial stress loading in different directions. The respective deformation histories are determined in the small strain-approximation, and demonstrate a significant influence of the initial load cycle on response from the underloaded configuration.

PREFACE

This report comprises two self-contained papers presented as Parts I and II. Part I, presented as the Fifth Periodic Report in May 1981, will be published shortly in Cold Regions Science and Technology. It is an investigation of viscoelastic solid relations of differential type which can be determined by constant stress and constant strain-rate responses of ice. Part II is an investigation of various classes of viscoelastic integral operator relations to see if improved and/or more tractable models of ice response can be constructed. Illustrations and conclusions particular to ice response are given. This paper will be submitted to Mechanics of Materials. Alternative representations to improve the correlation with ice data have been constructed in the current period and will be described in the next periodic report, February 1982. An associated paper will be submitted to Cold Regions Science and Technology.

PART I

Viscoelastic Solid Relations For The

Deformation Of Ice

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Abstract

A frame indifferent differential operator law relating stress, stress-rate, strain, and strain-rate is constructed to describe the qualitative features of both constant load and constant displacement rate response in uni-axial stress experiments. No lower order differential relation can describe both responses, but the two responses are not sufficient to determine the response coefficients of the relation. The jump relation is determined for a stress discontinuity applied in a general configuration. A simple initially isotropic model is proposed to investigate the effects of loading history and the anisotropy induced in a configuration following a loading-unloading cycle. Both uni-axial stress and shear cycles are treated, followed by uni-axial stress loading in different directions. The respective deformation histories are determined in the small strain-approximation, and demonstrate a significant influence of the initial load cycle on response from the unloaded configuration.

Introduction

Engineering application is mainly concerned with the mechanical response of polycrystalline ice to maintained stress up to strains of a few per cent, which includes the instantaneous elastic strain, primary decelerating creep, secondary or approximately steady creep, and some part of the accelerating tertiary creep. Figure 1 shows the typical strain and strain-rate responses to a constant uni-axial compressive nominal stress $\bar{\sigma}$ (force per unit initial cross-section) at constant temperature where the longitudinal engineering strain e is the contraction per unit initial length. Constant $\bar{\sigma}$ corresponds to constant axial load, which is a common test situation (Mellor 1980). Similarly, constant $\dot{e}$ corresponds to constant end displacement rate, a common alternative test configuration, and the associated stress response at constant temperature is shown in Fig. 2 as a function of strain $e = \dot{e}t$. Since the Young's modulus and bulk modulus of ice are of order $10^{10}\mathrm{Nm}^{-2}$ (Sinha 1978a, Michel 1978, Mellor 1980), a moderate stress of $10^{6}\mathrm{Nm}^{-2}$ induces an elastic strain of magnitude 10^{-4} compared with creep strains of magnitude 10^{-2} common in application, but elastic strains are retained in the present theory. The initial elastic strain is $e_e(\bar{\sigma}) = \bar{\sigma}/E_o$, linear in stress, where E_o is the Young's modulus at zero stress. The strain jump induced by a stress jump applied at an arbitrary stress level will be derived as a limit of smooth loading.

In Fig. 1(b), $\dot{e}_m(\bar{\sigma})$, $t_m(\bar{\sigma})$, are the stress dependent minimum strain-rate and time to minimum strain-rate, and

$e_m(\bar{\sigma})$ in Fig. 1(a) is the corresponding strain at the first inflexion point of the creep curve. The time $t_m(\bar{\sigma})$ increases significantly as $\bar{\sigma}$ decreases, and, more generally, so do the times to reach any strain on the creep curve; that is, the rate process decreases as $\bar{\sigma}$ decreases. Thus $\dot{e}_m(\bar{\sigma})$ is an increasing function of $\bar{\sigma}$, and we suppose also that the initial strain-rate $\dot{e}_o(\bar{\sigma})$ increases with $\bar{\sigma}$. The rate of decrease of length per unit current length, or compression strain-rate with respect to the current configuration, is given by

$$r = (1-e)^{-1}\dot{e} \ , \qquad e < 1 \ , \tag{1}$$

and the limit $e \to 1$ corresponds to the element length shrinking to zero. In glaciology it is commonly assumed that there is a long-time steady creep response $r_\infty(\sigma)$, where σ is the Cauchy stress, which is an increasing function and zero at $\sigma = 0$. With the later $\bar{\sigma}$, σ, e relation it then follows that

$$\dot{e} \to 0, \Rightarrow e \to 1, \quad \sigma \to 0, \quad r \to 0 \quad \text{as} \quad t \to \infty \ , \tag{2}$$

but this later part of the tertiary response with strains of order unity will not be reached in many engineering applications, particularly at low stress. We will focus on the small strain response shown by solid lines in Fig. 1 and Fig. 2.

At constant strain-rate $\dot{e}$, limited by $e < 1$, the maximum stress is $\bar{\sigma}_M(\dot{e})$, and the strain and time at maximum stress are $e_M(\dot{e})$, $t_M(\dot{e})$, where $e_M = \dot{e}t_M$. As the strain increases beyond e_M the stress $\bar{\sigma}$ relaxes, but must

subsequently increase as $e \to 1$. Mellor (1980) suggests that both e_m and e_M are approximately 0·01 over a wide range of stress and strain-rate respectively, and also that there are indications that the maximum stress $\bar{\sigma}_M(\dot{e})$ is the constant stress $\bar{\sigma}$ required to produce a minimum strain-rate $\dot{e}_m = \dot{e}$; that is

$$\dot{e}_m[\bar{\sigma}_M(\dot{e})] = \dot{e} \quad \text{and} \quad \bar{\sigma}_M[\dot{e}_m(\bar{\sigma})] = \bar{\sigma} . \tag{3}$$

The non-monotonic strain-rate response at constant stress and non-monotonic stress response at constant strain-rate, shown in Figs 1 and 2, have been modelled by a viscoelastic differential operator relation of fluid type (Morland and Spring 1981). The fluid relation is based on the strain-rate with respect to the current configuration and the Cauchy stress (force per unit current area). It is shown that the minimal form to describe both types of response requires strain-rate, strain-acceleration, stress, and stress-rate. There is no dependence on a reference configuration, and every configuration is necessarily isotropic. Thus, the model can neither describe an initial anisotropy which may arise in the ice formation, nor anisotropy of later configurations induced by the loading history. Also there is no elastic (jump) strain when a stress jump is applied. The strain-acceleration term is necessary for the non-viscous response to constant stress, and the stress-rate term is necessary for the non-uniform stress response at constant strain-rate.

We now present an analogous differential operator law of

solid type, with explicit dependence on the strain from a reference configuration. Again stress and stress-rate terms are necessary to describe the response at constant strain-rate, but strain and strain-rate terms are then sufficient to describe the non-uniform strain-rate response to constant stress. That is, at constant stress, the relation becomes a first order differential equation for strain, analogous to that for strain-rate in the fluid relation, and there is an appropriate non-uniform strain and strain-rate solution. The significant non-linearity of the response of ice to stress even at small strains means that the differential operator relation is highly non-linear. Linear viscoelastic models proposed by Nevel (1976) for low stress applications offer well developed methods of boundary-value problem solution, but are not an appropriate approximation. Sinha (1978a, 1978b) and Gold and Sinha (1980) have presented the results of constant uni-axial stress tests for short time, small strain, creep, and inferred an empirical relation for the axial strain in terms of stress and time explicitly. It is not shown, however, how this result can be related to a coordinate invariant tensor law necessary for combined stress loading, nor how the response to non-uniform stress history is constructed in the absence of linear superposition.

While the forms of the uni-axial responses determine the minimial set of terms required in a differential relation, they cannot determine the shape of the tensor relation which requires the response to tri-axial stress or combined axial and shear stress (Morland 1979). There are an infinity

of tensor relations which reduce to the same uni-axial response. In the absence of combined stress results we adopt a limited shape of tensor relation, with limited dependence of response coefficients on stress and strain, and examine its implications for uni-axial stress. It is related to families of creep curves for different constant stress, and families of stress curves for different constant strain-rates. In addition, the response to loading-unloading cycles is investigated, which demonstrates that a constant reduced stress response contains no new information about response coefficients, apart from providing limit functions at $\bar{\sigma} = 0$ when the stress is completely unloaded. However, measured reduced stress responses may provide consistency tests for the model.

We focus on a model which is isotropic in the reference configuration, and determine the differential equations describing combined shear and axial extensions in constant plane stress for the small strain approximation. An initial uni-axial load-unload cycle is analysed, followed by alternative uni-axial loads in two orthogonal directions. This demonstrates that if the initial load is repeated from the unloaded configuration, the response is different, which is one feature of the strain dependence absent in a fluid model. A distinct effect of the strain dependence is the induced anisotropy, or material asymmetry, in the unloaded configuration, demonstrated by the different responses to identical loads applied in the two directions. A shear load-unload cycle is also analysed, which, in the small strain approximation, is accompanied by negligible axial strains (a feature of

linear theory), and so induces an asymmetry in subsequent axial loading. However, it is shown that the shear strain can remain constant or decay significantly in subsequent axial loading, depending on the strain-tensor dependence.

Differential tensor relation

To obtain non-uniform strain-rate at constant stress, and non-uniform stress at constant strain-rate, in uni-axial stress, a differential operator relation must reduce in both cases to at least a first order differential equation for the respective response variable (Morland and Spring 1981). By analogy with the viscoelastic linear differential relation this implies that stress and stress-rate contributions are necessary, and that strain and strain-rate contributions are necessary, replacing the strain-rate and stress-acceleration combination of a fluid relation which is independent of strain.

We adopt the usual incompressibility approximation

$$\mathrm{tr}\,\underset{\sim}{D} = 0, \quad \det \underset{\sim}{F} = 1, \tag{4}$$

where the strain-rate $\underset{\sim}{D}$ with respect to the current configuration is the symmetric part of the spatial velocity gradient (and the rotation rate $\underset{\sim}{W}$ is the skew part), and $\underset{\sim}{F}$ is the deformation gradient. The Cauchy-Green strain tensors are

$$\underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^T, \quad \underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}, \tag{5}$$

with principal invariants

$$K_1 = \mathrm{tr}\underset{\sim}{B} = \mathrm{tr}\underset{\sim}{C}, \quad K_2 = \tfrac{1}{2}\{(\mathrm{tr}\underset{\sim}{B})^2 - \mathrm{tr}\underset{\sim}{B}^2\} = \tfrac{1}{2}\{(\mathrm{tr}\underset{\sim}{C})^2 - \mathrm{tr}\underset{\sim}{C}^2\},$$

$$K_3 = 1. \tag{6}$$

and the principal strain-rate invariants are

$$I_1 = \mathrm{tr}\,\underset{\sim}{D} \equiv 0, \quad I_2 = \tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{D}^2, \quad I_3 = \det \underset{\sim}{D}. \tag{7}$$

Incompressibility implies that only the stress deviator

$$\underset{\sim}{S} = \underset{\sim}{\sigma} - \frac{1}{3}(\mathrm{tr}\,\underset{\sim}{\sigma})\underset{\sim}{1}, \tag{8}$$

is determined by the deformation history, where $\underset{\sim}{\sigma}$ is the Cauchy stress. An associated frame-indifferent deviatoric stress-rate is

$$\underset{\sim}{S}^{(1)} = \dot{\underset{\sim}{S}} + \underset{\sim}{S}(\underset{\sim}{D} + \underset{\sim}{W}) + (\underset{\sim}{D} - \underset{\sim}{W})\underset{\sim}{S}, \tag{9}$$

where a superposed dot denotes material time derivative. The principal deviatoric stress invariants are

$$J_1 = \mathrm{tr}\,\underset{\sim}{S} \equiv 0, \quad J_2 = \tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{S}^2, \quad J_3 = \det \underset{\sim}{S}. \tag{10}$$

With the further simplifications that the stress and stress-rate tensors enter in a linear combination, and that the strain and strain-rate tensors enter as separate terms, then the frame-indifferent differential relation described above takes the form

$$\underset{\sim}{S} + \psi[\underset{\sim}{S}^{(1)} - \frac{2}{3}(\mathrm{tr}\,\underset{\sim}{S}\underset{\sim}{D})\underset{\sim}{1}]$$

$$[\underset{\sim}{F}\,\underset{\sim}{f}\,(\underset{\sim}{C})\underset{\sim}{F}^T - \frac{1}{3}(\mathrm{tr}\,\underset{\sim}{C}\,\underset{\sim}{f})\underset{\sim}{1}] + \phi_1\underset{\sim}{D} + \phi_2[\underset{\sim}{D}^2 - \frac{2}{3}I_2\underset{\sim}{1}]. \tag{11}$$

Here $\underset{\sim}{f}$ is an arbitrary symmetric tensor function of symmetric tensor argument, and the response coefficients ψ, ϕ_1, ϕ_2 are

functions of the invariants K_1, K_2, I_2, I_3, and their rates. Dependence on stress and stress-rate is therefore not generally linear, and coupled dependence on stress, strain, and their rates can arise through invariant products. We will present the analysis for a reference configuration which is an isotropic configuration, and demonstrate the anisotropic response from configurations reached by different loadings. The isotropic reduction of (11) is given by requiring the F dependence to be an isotropic function of B, already imposed for the dependence on D by frame-indifference, hence

$$S + \eta[S^{(1)} - \frac{2}{3}\mathrm{tr}(S D) 1] = \phi_1 D + \phi_2[D^2 - \frac{2}{3}I_2 1]$$

$$+ \omega_1[B - \frac{1}{3}K_1 1] + \omega_2[B^2 - \frac{1}{3}(K_1^2 - 2K_2) 1] , \qquad (12)$$

where the additional response coefficients ω_1, ω_2 are also functions of invariants.

While dependence on the strain tensor is absent if $\omega_1 = \omega_2 = 0$, dependence on strain can still enter through dependence of η, ϕ_1, ϕ_2 on the strain invariants K_1, K_2. Furthermore, such dependence on invariants K_1, K_2 only is sufficient to induce anisotropic response from subsequent configurations. Consider a second reference configuration reached from the first by a deformation F_o, and let $\tilde{F}$ denote the deformation from the second reference configuration. Then

$$F = \tilde{F} F_o , \quad B = \tilde{F} B_o \tilde{F}^T , \qquad (13)$$

$$K_1 = \mathrm{tr}(\tilde{C} B_o) , \quad K_2 = \frac{1}{2}(K_1^2 - \mathrm{tr}(\tilde{C} B_o \tilde{C} B_o))$$

while the strain invariants from the second reference configuration are

$$\tilde{K}_1 = \mathrm{tr}\,\tilde{\underset{\sim}{C}}, \quad \tilde{K}_2 = \tfrac{1}{2}(\tilde{K}_1^2 - \mathrm{tr}\,\tilde{C}^2) \quad . \tag{14}$$

Thus, unless $\underset{\sim}{B}_o = b\,\underset{\sim}{1}$, when the change of configuration is simply an isotropic dilatation (or compression), $\tilde{K}_1$, $\tilde{K}_2$ cannot be expressed in terms of the invariants K_1, K_2 alone, but depend on the strain tensor C explicitly. That is, response functions ψ, ϕ_1, ϕ_2, of K_1, K_2 exhibit explicit dependence on the strain tensor C from a second reference configuration in general, not just on the invariants $\tilde{K}_1$, $\tilde{K}_2$, so the second configuration is not isotropic when $\omega_1 = \omega_2 = 0$. Such anisotropy will be demonstrated by example.

Jump relations

It remains to complement the differential relation (12) by jump conditions which determine the strain jump when a stress jump is applied in any configuration during loading. If the jump relation is the limit of a sequence of smooth rapid changes governed by the differential relation (12), then the response coefficients ψ, ϕ_1, ϕ_2, ω_1, ω_2 must remain bounded as stress-rates and strain-rates become infinite, which is a restriction on the dependence on $\dot{J}_2$, $\dot{J}_3$, and $\dot{K}_1$, $\dot{K}_2$, or I_2, I_3. Further, since $\int \underset{\sim}{D}^2 dt$ is unbounded as the time interval approaches zero, ϕ_2 must approach zero sufficiently fast as stress and strain-rates become infinite, which implies that a non-zero ϕ_2 must depend on the invariant rates. Since uni-axial response does not seperate the ϕ_1 and ϕ_2 terms, andsince we subsequently analyse models which

exclude dependence of the response coefficients on the invariant rates, we henceforth adopt the simplification

$$\phi_2 = 0 \,. \tag{15}$$

Suppose the deformation is changed instantaneously at time t from $\mathbf{F}_o(t)$ to $\mathbf{F}$ by a stress change from $\mathbf{S}_o(t)$ to $\mathbf{S}$. Since such elastic strains are infinitesimal, and assuming there is no allied rigid rotation jump, we will adopt the linear approximations

$$\begin{aligned} &\mathbf{W} = \mathbf{0}, \qquad \mathbf{F} + (\mathbf{1} + \boldsymbol{\varepsilon})\,\mathbf{F}_o\,, \\ &\mathbf{D}_o = \dot{\mathbf{F}}_o\mathbf{F}_o^{-1}, \qquad \mathbf{D} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{D}_o + \dot{\boldsymbol{\varepsilon}}, \end{aligned} \tag{16}$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain from the starting configuration with deformation $\mathbf{F}_o$. That is, $\boldsymbol{\varepsilon}$ is the strain jump which will depend on the stress jump, the starting stress $\mathbf{S}_o$, and starting deformation $\mathbf{F}_o$. Let

$$\begin{aligned} &\boldsymbol{\varepsilon} = \mathbf{h}\,(\mathbf{S}, \mathbf{S}_o, \mathbf{F}_o), \qquad \frac{\partial \mathbf{h}}{\partial \mathbf{S}} = \mathbf{g}\,(\mathbf{S}, \mathbf{S}_o, \mathbf{F}_o), \\ &\dot{\boldsymbol{\varepsilon}} = \frac{\partial \mathbf{h}}{\partial \mathbf{S}}\,\dot{\mathbf{S}} + \frac{\partial \mathbf{h}}{\partial \mathbf{S}_o}\,\dot{\mathbf{S}}_o + \frac{\partial \mathbf{h}}{\partial \mathbf{F}_o}\,\dot{\mathbf{F}}_o\,, \simeq \mathbf{g}\,\dot{\mathbf{S}}\,, \end{aligned} \tag{17}$$

in the limit process, since $\dot{\mathbf{S}}$ becomes infinite while $\mathbf{S}_o$, $\mathbf{F}_o$, and the respective $\mathbf{h}$ derivatives are supposed bounded. Integrating (12) over a vanishing time interval, and assuming the jump relation $\mathbf{h}$ holds continuously through the change, gives

$$\begin{aligned} &\int_{\mathbf{S}_o}^{\mathbf{S}} \psi\left[dS_{ij} + S_{ik}g_{k\ell}dS_{\ell j} - \tfrac{2}{3}\delta_{ij}S_{pq}g_{qr}dS_{rp}\right] \\ &\qquad = \int_{\mathbf{S}_o}^{\mathbf{S}} \phi_1 g_{ik}dS_{kj}, \end{aligned} \tag{18}$$

where ψ, ϕ_1 are evaluated in the infinite rate limit (if dependent on invariant rates) and at $\underset{\sim}{F} = \underset{\sim}{F}_o$, but retain their dependence on $\underset{\sim}{S}$. Given $\underset{\sim}{S}_o$, $\underset{\sim}{F}_o$, the six integral relations (18) hold for arbitrary $\underset{\sim}{S}$ and determine the six components $g_{ij}(\underset{\sim}{S}, \underset{\sim}{S}_o, \underset{\sim}{F}_o)$, and hence $\underset{\sim}{h}(\underset{\sim}{S}, \underset{\sim}{S}_o, \underset{\sim}{F}_o)$ by (17) and the initial condition $\underset{\sim}{h}(\underset{\sim}{S}_o, \underset{\sim}{S}_o, \underset{\sim}{F}_o) = \underset{\sim}{0}$.

Now $||\underset{\sim}{g}||^{-1}$ represents an elastic modulus magnitude, so that $||\underset{\sim}{S}||.||\underset{\sim}{g}|| \sim ||\underset{\sim}{\varepsilon}|| << 1$, so consistent with the infinitesimal elastic strain approximation, (18) becomes

$$\int_{\underset{\sim}{S}}^{\underset{\sim}{S}} (\psi dS_{ij} - \phi_1 g_{ik} dS_{kj}) = 0 \,. \qquad (19)$$

A solution of (19) is illustrated for uni-axial stress loading.

Uni-axial stress

For incompressible material the nominal stress $\bar{\underset{\sim}{\sigma}}$ and Cauchy stress $\underset{\sim}{\sigma}$ are related by (Chadwick 1976)

$$\underset{\sim}{\sigma} = \underset{\sim}{F}\,\underset{\sim}{\sigma} \,. \qquad (20)$$

In uni-axial compressive stress $\sigma_{11} = -\sigma < 0$, other $\sigma_{ij} = 0$, with corresponding principal stretches $\lambda_{11} = \lambda < 1$, $\lambda_{22} = \lambda_{33} = \lambda^{-\frac{1}{2}}$, and nominal stress $\bar{\sigma}_{11} = -\bar{\sigma}$, other $\bar{\sigma}_{ij} = 0$,

$$\sigma = \lambda\bar{\sigma}, \qquad \dot{\sigma} = \lambda\dot{\bar{\sigma}} + \dot{\lambda}\bar{\sigma} . \tag{21}$$

Also

$$\lambda = 1-e , \qquad \dot{\lambda} = -\dot{e} , \tag{22}$$

where e is the axial contraction per unit initial length, and from (5) $B = C$ with $B_{11} = \lambda^2$, $B_{22} = B_{33} = \lambda^{-1}$, $B_{ij} = 0$ $(i \neq j)$, so from (6)

$$K_1 = (1-e)^2 + 2(1-e)^{-1}, \; K_2 = 2(1-e) + (1-e)^{-2} , \tag{23}$$

or, if terms of order e^3 are neglected,

$$K_1 = K_2 = 3(1+e^2) . \tag{24}$$

The strain-rate $\underset{\sim}{D}$ has components $D_{11} = -r$, $D_{22} = D_{33} = \frac{1}{2} r$, related to $\dot{e}$ by (1), and invariants

$$I_2 = \frac{3}{4} r^2 , \quad I_3 = -\frac{1}{4} r^3 . \tag{25}$$

The deviatoric stress has components $S_{11} = -\frac{2}{3}\sigma$, $S_{22} = S_{33} = \frac{1}{3}\sigma$, other $S_{ij} = 0$, and invariants

$$J_2 = \frac{1}{3}\sigma^2 = \frac{1}{3}(1-e)^2\bar{\sigma}^2 , \; J_3 = -\frac{2}{27}\sigma^3 = -\frac{2}{27}(1-e)^3\bar{\sigma}^3 . \tag{26}$$

We will restrict the analysis to models in which the response coefficients depend only on the stress and strain invariants J_2, J_3, K_1, K_2, and not on rates. In uni-axial

stress they become functions of two variables $\bar{\sigma}$ and e, and explicit notation for the reduced dependence is useful; thus

$$\psi = \hat{\psi}(\bar{\sigma},e),\ \phi_1 = \hat{\phi}_1(\bar{\sigma},e),\ \omega_1 = \hat{\omega}_1(\bar{\sigma},e),\ \omega_2 = \hat{\omega}_2(\bar{\sigma},e). \quad (27)$$

Given dependence on the invariants determines the reduced functions, but the latter cannot distinguish dependence on J_2 and J_3, and on K_1 and K_2. That is, there are an infinity of response coefficient sets compatible with the same uni-axial stress response, and the response for two independent stresses and two independent strains is required to construct the general loading relation for an incompressible material. For illustration of shear response we interpret a model dependence on $\bar{\sigma}, e$, in terms of J_2, K_1. Clearly a non-symmetric response in compression and tension would require dependence on both even and odd invariants.

Each principal component of (12) gives the same relation, which, using the restriction (17), is

$$(1-e)^3\bar{\sigma} + \hat{\psi}(1-e)^2[(1-e)\dot{\bar{\sigma}} - 2\dot{e}\bar{\sigma}] = \tfrac{3}{2}\hat{\phi}_1(1-e)\dot{e} + \hat{\omega}e\ , \quad (28)$$

where

$$\begin{aligned}\hat{\omega}e &= \hat{\omega}_1(1-e)[1-(1-e)^3] + \hat{\omega}_2[1-(1-e)^6], \\ &= (3\hat{\omega}_1 + 6\hat{\omega}_2)e + O(e^2).\end{aligned} \quad (29)$$

Only the combination $\hat{\omega}$ of $\hat{\omega}_1, \hat{\omega}_2$, occurs, so that uni-axial stress response cannot distinguish the contribution of the ω_1 and ω_2 terms in (12). With the restricted

dependence (27) the response coefficients are necessarily **bounded** as rates become infinite, and the jump relation **for** a **stress jump** $\bar{\sigma}_o$ to $\bar{\sigma}$ with corresponding **strain jump** e_o to $e_o + \hat{h}(\bar{\sigma},\bar{\sigma}_o,e_o)$ is obtained by integrating (28) **with** respect to time over a vanishing time interval

$$\int_{\bar{\sigma}_o}^{\bar{\sigma}} [\hat{\psi}(1-e)^3 - \{\tfrac{3}{2}\hat{\phi}_1(1-e) + 2\bar{\sigma}\hat{\psi}(1-e)^2\}\frac{\partial\hat{h}}{\partial\bar{\sigma}}]\,d\bar{\sigma} = 0. \qquad (30)$$

Since (30) holds for all $\bar{\sigma}$ for given $\bar{\sigma}_o, e_o$, the integrand **vanishes** to give a first order differential equation for **the** strain-jump $\hat{h}(\bar{\sigma},\bar{\sigma}_o,e_o)$, namely

$$\{2\bar{\sigma}(1-e_o-\hat{h})\hat{\psi}(\bar{\sigma},e_o+\hat{h}) + \tfrac{3}{2}\hat{\phi}_1(\bar{\sigma},e_o+\hat{h})\}\frac{\partial\hat{h}}{\partial\bar{\sigma}}$$

$$= (1-e_o-\hat{h})^2\,\hat{\psi}(\bar{\sigma},e_o+\hat{h}), \qquad \hat{h}(\bar{\sigma}_o,\bar{\sigma}_o,e_o) = 0. \qquad (31)$$

Applying the infinitesimal strain jump approximation $|\hat{h}| \ll 1$, $|\bar{\sigma}\frac{\partial\hat{h}}{\partial\bar{\sigma}}| \ll 1$, and evaluating the response **coefficients** at $\underset{\sim}{F}_o$, that is $\hat{\psi}$, $\hat{\phi}_1$ at $e = e_o$ (even when $e_o = 0$), **gives** the simpler equation

$$\tfrac{3}{2}\hat{\phi}_1(\bar{\sigma},e_o)\frac{\partial\hat{h}}{\partial\bar{\sigma}} = (1-e_o)^2\,\hat{\psi}(\bar{\sigma},e_o)\;, \qquad (32)$$

in which $\hat{h}$ **does not** appear in arguments of the **response coefficients.** **While** e_o may be neglected in **comparison with unity at small total** strain, the $\hat{\psi}$, $\hat{\phi}_1$ **dependence on** e_o **may be significant.** The initial strain **jump is** $e_e(\bar{\sigma}) = \hat{h}(\bar{\sigma},0,0)$, $\sim 2\bar{\sigma}\hat{\psi}(0,0)/3\hat{\phi}_1(0,0)$ as $\bar{\sigma} \to 0$. **The condition (32) is given directly by (19) since** $\hat{h}(\bar{\sigma},\bar{\sigma}_o,e_o)$ $= -(1-e_o)\,h_{11}(\underset{\sim}{S}, \underset{\sim}{S}_o, \underset{\sim}{F}_o)$.

At constant load, $\dot{\bar{\sigma}} = 0$, (28) reduces to an explicit expression for $\dot{e}$ in terms of $\bar{\sigma}$ and e; that is, a first order differential equation for e. Since $e(t)$ is monotonic for $\bar{\sigma} > 0$, Fig. 1a, the strain-rate response shown in Fig. 1b can be expressed as a function of e instead of t, and hence the family of response curves for different constant $\bar{\sigma}$ determine a relation $\dot{e} = F(\bar{\sigma},e)$ for $e \geq e_e(\bar{\sigma})$, $\bar{\sigma} > 0$. By (28),

$$\dot{\bar{\sigma}} = 0 : \dot{e} = F(\bar{\sigma},e) = \frac{(1-e)^3 \text{ [illegible] } e}{\frac{3}{2}\hat{\gamma}_1(1-e) + 2\text{[illegible]}(1-e)^2}, \quad e(0) = e_e(\bar{\sigma}). \qquad (33)$$

At constant strain-rate $\dot{e} = w$, the typical stress-strain curve is shown in Fig. 2. The peak stress $\bar{\sigma}_M$ increases with w, but it was noted that the corresponding strain e_M is approximately constant ($\approx 0{\cdot}01$), so it is reasonable to assume that at each strain e (at least in the small strain range) the stress increases with w; that is, the curves for different w do not intersect. Hence at each value of e there is a monotonic $\bar{\sigma} - w$ relation, which can therefore be inverted:

$$\dot{e} = w = \text{constant:} \quad \bar{\sigma} = G(w,e), \; w = W(\bar{\sigma},e). \qquad (34)$$

Thus the family of response curves $G(w,e)$ lead to a family of generalised Young's moduli $(\partial G/\partial e)_w$ which can be expressed as a function $E(\bar{\sigma},e)$. From (28), and using the defenitions (33) and (34) of F and W:

$$\dot{e} = w: \left.\frac{\partial\bar{\sigma}}{\partial e}\right|_{w} = E(\bar{\sigma},e) = \left\{\frac{3\hat{\phi}_1}{2\hat{\psi}(1-e)^3} + \frac{2\bar{\sigma}}{1-e}\right\}\left\{1 - \frac{F}{W}\right\}. \tag{35}$$

The initial Young's modulus $(w \to \infty,\ \bar{\sigma} \to 0,\ e \to 0)$ is

$$E_o = E(0,0) = \frac{3\hat{\phi}_1(0,0)}{2\hat{\psi}(0,0)}\ , \tag{36}$$

since $F(0,0) = 0$, which is independent of w. This is a consequence of assuming rate independent response coefficients, but is a good approximation over a practical range of w with the value $E_o = 9 \times 10^9\,\mathrm{Nm}^{-2}$ (Michel 1978). While $E_o \gg \bar{\sigma}$, $E(\bar{\sigma}_M, e_M) = 0$.

Given $E(\bar{\sigma},e)$, $W(\bar{\sigma},e)$, and $F(\bar{\sigma},e)$, from the uni-axial stress tests, the constant strain-rate relation (35) determines the ratio $\hat{\phi}_1/\hat{\psi}$, then the constant stress relation (33) determines the combination $[(1-e)^3\,\bar{\sigma} - \hat{\omega}e]/\hat{\psi}$, so that a further uni-axial result is required to determine both $\hat{\omega}$ and $\hat{\psi}$. In the analogous viscoelastic fluid model the equivalent results to (33) and (35) were not completely independent, and the adopted model would not be compatible with arbitrary F and W. If at a time t_1 during constant stress $\bar{\sigma}$ loading, the stress is partially unloaded to $\bar{\sigma}_u$ (> 0) and then again maintained constant, the strain undergoes an elastic decrease to $e(t_1+)$ governed by the jump relation (32), and then for $t > t_1$ is governed by the differential equation (33) with $\bar{\sigma} \to \bar{\sigma}_u$ and $e(t_1+)$ as starting value. Given that $F(\bar{\sigma},e)$ has been determined over the necessary range of $(\bar{\sigma},e)$ by families of loading

responses, then $F(\bar{\sigma}_u, e)$ is known and the response in $t > t_1$ is determined; that is, the same combination $[(1-e)^3\,\bar{\sigma} - \hat{\omega}e]/\hat{\psi}$ arises, and the unloading response does not separate $\hat{\omega}, \hat{\psi}$.

However, in the complete unloading case, $\bar{\sigma}_u = 0$,

$$t > t_1 : \quad \dot{e} = F(0,e) = - \frac{\hat{\omega}(0,e)e}{3\hat{\psi}_1(0,e)(1-e)}, \tag{37}$$

which determines $F(0,e)$, hence $\hat{\omega}(0,e)/\hat{\psi}_1(0,e)$, not determined by initial loading $\bar{\sigma} > 0$ (when $e > 0$). We are not aware that details of the response to complete unloading have been determined. Three possible general situations are illustrated in Fig. 3; (i) no relaxation ($\dot{e} = 0$), (ii) partial relaxation ($\dot{e} < 0$, $e \to e_\infty > 0$), (iii) complete relaxation ($\dot{e} < 0$, $e \to 0$), based on the assumptions $\hat{\psi}_1(0,e) > 0$, $\hat{\omega}(0,e) \geq 0$. It is also feasible that creep continues ($\dot{e} > 0$), either indefinitely to a limit or to some maximum followed by relaxation, for example if $\hat{\omega}(0,e) < 0$ for all $0 < e < 1$, or for some e before changing sign as e increases. This does not appear to be a smooth limit of the response coefficients for $\bar{\sigma} > 0$. Case (i) is equivalent to $\hat{\omega}(0,e) = 0$ at $e = e(t_1+)$, or $\hat{\omega}(0,e) \equiv 0$ if there is no relaxation from any pre-loaded configuration.

Idealised response

The reduced response coefficients are defined by the functions $F(\bar{\sigma},e)$, $E(\bar{\sigma},e)$ representing families of responses illustrated in Fig. 1 and Fig. 2, together with an extra

assumption to uncouple [illegible] and . We will now construct a simple model which has the appropriate qualitative features. First introduce a reduced time $\hat{t} = t/t_m(\bar{\sigma})$ and suppose that $(de/d\hat{t})$ at constant $\bar{\sigma}$, and $e_m(\bar{\sigma})$, are independent of $\bar{\sigma}$. Setting

$$e_m = 0{\cdot}01, \quad \left.\frac{de}{d\hat{t}}\right|_{\bar{\sigma}} = t_m(\bar{\sigma})\dot{e}(\bar{\sigma},e) = kf(e), \; f(e_m) = 1, \tag{38}$$

then k is a constant and

$$k = t_m(\bar{\sigma})\dot{e}_m(\bar{\sigma}), \; F(\bar{\sigma},e) = \dot{e} = \dot{e}_m(\bar{\sigma})f(e), \quad t_m(\bar{\sigma}) = \dot{e}_m^{-1}(\bar{\sigma}) \int_0^{e_m} f^{-1}(e')de', \tag{39}$$

where the universal function $f(e)$ has the same shape as $F(\bar{\sigma},e)$ for each constant $\bar{\sigma}$. We take the Colbeck and Evans (1973) power law for the minimum strain-rate:

$$\dot{e}_m(\bar{\sigma}) = 0{\cdot}21\,\bar{\sigma} + 0{\cdot}14\,\bar{\sigma}^3 + 0{\cdot}055\,\bar{\sigma}^5, \quad \bar{\sigma} \text{ in } 10^5\,\mathrm{Nm}^{-2}, \tag{40}$$

where the stress and time units are $10^5\,\mathrm{Nm}^{-2}$ and one year $a = 3{\cdot}15 \times 10^7$ s respectively. Thus $F(0,e) = 0$, and so there is no relaxation after complete unloading, and case (i) of Fig. 3 always applies. Also (39) extends F into $0 \le e < e_e(\bar{\sigma})$.

The main features of $f(e)$ shown in Fig. 1 can be represented by

$$f(e) = R + Be^{-be} + Ce^{-ce} \tag{41}$$

where R is the ratio of the strain-rate at "large e" (long time) to the minimum strain-rate (here independent of $\bar{\sigma}$).

B,b,C,c are determined by (38), $f'(e_m) = 0$, $f''(e_i) = 0$, where e_i is the strain at inflexion, and $f(0) = f_o$, where f_o is the ratio of the initial strain-rate to the minimum strain-rate (here independent of $\bar{\sigma}$). Assuming R = 2·5, suggested by Steinemann's (1958) data, and making the arbitrary choices e_i = [illegible], f_o = 6·[illegible]7, then

$$R = 2\cdot 5,\ B = -\ 5\cdot 6,\ b = [illegible],\ C = 11\cdot 7,\ c = 274\cdot 7\ . \tag{42}$$

The reduced time $\hat{t}$ is related to the strain e by

$$\int_0^e f^{-1}(e')de' = \dot{e}_m(\bar{\sigma})t = \hat{t}\int_0^{e_m} f^{-1}(e')de', = 0.00552\hat{t} \tag{43}$$

for this model.

Figure 4 shows the above f as a function of e/e_m and as a function of $\hat{t}$, and the associated strain-reduced time curve $F(\bar{\sigma},e)$ is now prescribed.

It is convenient to express the generalised Young's modulus $E(\bar{\sigma},e)$ in the factored form

$$E(\bar{\sigma},e) = \hat{E}\left\{1 - \frac{F}{W}\right\},\ \hat{E} = \frac{3[illegible]}{2\hat{C}(1-e)^2} + \frac{2\bar{\sigma}}{1-e}\ , \tag{44}$$

where the factor $\hat{E}(\bar{\sigma},e)$ is common to all strain-rates w, and $\hat{E}(0,0) = E_o$. We now examine the stress-strain response at constant strain-rate, $\bar{\sigma} = G(w,e)$, under three different models for $\hat{E}$, all independent of $\bar{\sigma}$:

$$\text{(i)}\ \hat{E} \equiv E_o,\quad \text{(ii)}\ \hat{E} = E_o e^{-1600e} + E_1,\ \text{(iii)}\ \hat{E} = E_o e^{-800e} + E_1 \tag{45}$$

$$E_o = 9 \times 10^4 \cdot 10^5 Nm^{-2},\ E_1 = 20 \times 10^5 Nm^{-2}.$$

In the exponential decay cases (ii) and (iii), the constant

E_1 ($<< E_o$) ensures that [illegible] $(1-e)^2$ [illegible] remains positive at strains of order 0·05, assuming [illegible] has relaxed below $10^5 Nm^{-2}$, so that $\partial h/\partial \bar{\sigma} > 0$ in the jump relation (32). At constant $\dot{e} = w$, by (35), (44), [illegible],

$$\frac{d\bar{\sigma}}{de} = \hat{E}(e)\left\{1 - \frac{F(\bar{\sigma},e)}{w}\right\}, \qquad [illegible] \text{ at } e = 0, \tag{46}$$

which has been integrated numerically for the above F and different $\hat{E}(e)$, yielding the stress-strain curves shown in Fig. 5 for four values of w. The exponential decay of case (ii) shows little difference from the case $\hat{E} \equiv E_o$, in $e < 0{\cdot}06$, but the more rapid decay of case (iii) fails to show a required peak at [illegible] e_m.

Both case (i) and (ii) show the required qualitative response, which is relatively insensitive to this variation of $\hat{E}(e)$, so the constant stress function $F(\bar{\sigma},e)$ also controls the constant strain-rate response. Since $F(\bar{\sigma},e)$ defines a family of non-intersecting strain-rate functions of e with $(\partial F/\partial\bar{\sigma}) > 0$, there is a unique curve $\bar{\sigma} = \hat{\sigma}(w,e)$ for each $w > 0$ on which $F = W$, and $F \gtrless W$ for $\bar{\sigma} \gtrless \hat{\sigma}$. Thus the peak point $[e_M, \bar{\sigma}_M = G(w,e_m)]$ occurs on $\bar{\sigma} = \hat{\sigma}(w,e)$, since $(\partial G/\partial e) = 0$ at $e = e_M$, and $(\partial G/\partial e) \gtrless 0$ as $\bar{\sigma} \lessgtr \hat{\sigma}$. The curves $\bar{\sigma} = \hat{\sigma}(w,e)$ are not distinguishable from the solutions $\bar{\sigma} = G(w,e)$ for $\hat{E} \equiv E_o$ shown in Fig. 5. In all cases the slope approaches E_o as $e \to 0$ and the initial stress rise appears discontinuous on the figure scales. Because the variation between curves (i) and (ii) has little influence on the modulus $\hat{E}(\bar{\sigma},e)$, the case $\hat{E} \equiv E_o$ is adopted for further illustration.

and (40)

Since $F(0,e) = 0$ by $(39)_{ii}$, implying no relaxation after unloading, the extra assumption on $\hat{\omega}$, $\hat{\psi}$ must give $\hat{\omega}(0,e) = 0$. For illustration we choose

$$e\hat{\omega}(\bar{\sigma},e) = \alpha(1-e)^3\bar{\sigma} , \quad 0 < \alpha < 1. \tag{47}$$

When $\alpha = 0$, $\hat{\omega} \equiv 0$ and only strain invariants enter the response, while for α order unity the strain tensor term $\hat{\omega}e$ contributes equally with the $\bar{\sigma}$ term to the strain-rate (33). Now, by (33), (39) and (44),

$$\hat{\psi} = \frac{(1-\alpha)\bar{\sigma}}{E_o\dot{e}_m(\bar{\sigma})f(e)} , \quad \hat{\phi}_1 = \frac{2(1-\alpha)(1-e)^2\bar{\sigma}}{3\dot{e}_m(\bar{\sigma})f(e)}\left[1 - \frac{2\bar{\sigma}}{E_o(1-e)}\right], \tag{48}$$

both positive and bounded as $\bar{\sigma} \to 0$ and $e \to 0$. The initial elastic strain, setting $e_o = 0$ in (32), is $e_e(\bar{\sigma}) = \bar{\sigma}/E_o$. To obtain response coefficients $\hat{\psi}$, $\hat{\phi}_1$, ω_1, ω_2, as functions of the stress and strain invariants for multi-axial and shear load illustrations, we assume dependence is on J_2 and K_1 only, and use the small strain approximations $e << 1$. From (24) and (26),

$$e = \left(\frac{K_1 - 3}{3}\right)^{\frac{1}{2}} , \quad \bar{\sigma} = (3J_2)^{\frac{1}{2}} , \tag{49}$$

then (40) becomes

$$\dot{e}_m(\bar{\sigma}) = \bar{\sigma}(0{\cdot}21 + 0{\cdot}42J_2 + 0{\cdot}495J_2^2) = \bar{\sigma}\Omega(J_2). \tag{50}$$

Choosing $\hat{\omega}_2 = 0$ in the definition (29) of $\hat{\omega}$ and neglecting e compared to unity, the model (47), (48) gives

$$\omega_1 = \alpha\left(\frac{J_2}{K_1-3}\right)^{\frac{1}{2}}, \quad E_o\psi = \frac{1-\iota}{\Omega(J_2)\, f\left[\left(\frac{K_1-3}{3}\right)^{\frac{1}{2}}\right]}, \tag{51}$$

$$\phi_1 = \frac{2}{3}E_o\psi - 4(J_2/3)^{\frac{1}{2}},$$

where $f(e)$ is defined by (41), (42), and the simplified relation (12), with (15), is

$$\underset{\sim}{S} + \psi[\underset{\sim}{S}^{(1)} - \frac{2}{3}\mathrm{tr}(\underset{\sim}{S}\,\underset{\sim}{D})\,\underset{\sim}{1}] = \phi_1 \underset{\sim}{D} + \omega_1[\underset{\sim}{B} - \frac{1}{3}K_1\underset{\sim}{1}] \quad . \tag{52}$$

Loading cycles and induced anisotropy

To demonstrate the anisotropy induced in configurations reached by a stress-loading-unloading cycle from an initial isotropic configuration, we will determine and compare strain responses predicted by the relations (51), (52) for $\alpha = 0$ and $\alpha \neq 0$ when the initial cycle is on uni-axial stress and when the initial cycle is on shear stress. In both cases the initial constant stress is applied over a time interval $(0,t_1)$, where $t = t_1$ is in the tertiary creep range, and unloaded completely at $t = t_1$ for a time interval (t_1,t_2). Since the model (51), (52) is non-relaxing, the strain remains constant over (t_1,t_2). At time t_2, two alternative reloading situations are considered to compare the induced directional properties of the configuration at time t_2; namely, the same constant uni-axial stress for $t > t_2$ applied (separately) in two orthogonal directions. Table 1 shows these four loading

histories, further illustrated by the section (a) of Figs 6 - 8. Non-symmetry of the nominal stress $\bar{\sigma}$ implies $\bar{\sigma}_{12} \neq \bar{\sigma}_{21}$ in shear, and zero normal traction on the sheared faces X_1 = constant requires one normal nominal stress non-zero, here $\bar{\sigma}_{11}$.

$0 < t < t_1$:	$\bar{\sigma}_{11} = -\bar{\sigma}_o$ other $\bar{\sigma}_{ij} = 0$		$\bar{\sigma}_{21} = \bar{\tau}_o$, $\bar{\sigma}_{12}, \bar{\sigma}_{11} \propto \bar{\tau}_o$ other $\bar{\sigma}_{ij} = 0$
$t_1 < t < t_2$:		$\bar{\sigma} = 0$	
	(a)	or	(b)
$t > t_2$:	$\bar{\sigma}_{11} = -\bar{\sigma}_o$ other $\bar{\sigma}_{ij} = 0$		$\bar{\sigma}_{22} = -\bar{\sigma}_o$ other $\bar{\sigma}_{ij} = 0$

Table 1. Loading-unloading-reloading cycles

The above examples involve only plane stress in OX_1X_2, and the principal stretch along OX_3 is given by the incompressibility condition. In addition, only the (1,1), (1,2) (2,2) components of (52) are independent equations, so only OX_1X_2 components of the various tensors are displayed. For

combined stretches λ_1 along OX_1, λ_2 along OX_2 and shear $\tan^{-1}(\frac{\gamma}{\lambda_2})$ between OX_1 and OX_2, the general forms are

$$\underset{\sim}{F} = \begin{pmatrix} \lambda_1 & \gamma \\ 0 & \lambda_2 \end{pmatrix}, \quad F_{33} = (\lambda_1 \lambda_2)^{-1}, \quad \underset{\sim}{\bar{\sigma}} = \begin{pmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{12} \\ \bar{\sigma}_{21} & \bar{\sigma}_{22} \end{pmatrix},$$

$$\underset{\sim}{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 \bar{\sigma}_{11} + \gamma \bar{\sigma}_{21} & \lambda_1 \bar{\sigma}_{12} + \gamma \bar{\sigma}_{22} \\ \lambda_2 \bar{\sigma}_{21} & \lambda_2 \bar{\sigma}_{22} \end{pmatrix}, \tag{53}$$

using (20), where λ_1, λ_2 are principal stretches only if the shear $\gamma = 0$. By symmetry of the Cauchy stress

$$\bar{\sigma}_{12} = \frac{\lambda_2}{\lambda_1} \bar{\sigma}_{21} - \frac{\gamma}{\lambda_1} \bar{\sigma}_{22},$$

$$= \begin{cases} 0 & \bar{\sigma}_{21} = 0, \ \gamma = 0 \\ \frac{\lambda_2}{\lambda_1} \bar{\sigma}_{21} & \bar{\sigma}_{22} = \bar{\sigma}_{33} = 0. \end{cases} \tag{54}$$

The latter situation is adopted for the shear cycle, so that no normal load is applied in conjunction with the nominal shear stress $\bar{\sigma}_{21}$ on the loaded planes X_2 = constant (x_2 = constant). Surface traction is given by (Chadwick 1976)

$$\underline{t}_{(\underline{n})} da = \underset{\sim}{\sigma}^T \underline{n}\, da = \underset{\sim}{\bar{\sigma}}^T \underline{N} dA, \tag{55}$$

where $\underline{n}\, da$ and $\underline{N}\, dA$ are the current and initial elements of vector area. On a sheared face X_1 = constant,

$$\underset{\sim}{N} = (1,0)\,,\quad \underset{\sim}{n} = \left(1 + \frac{\gamma^2}{\lambda_2^2}\right)^{-1/2} \left(1, -\frac{\gamma}{\lambda_2}\right)\,,$$

$$\bar{\underset{\sim}{\sigma}}^T \underset{\sim}{N}\cdot\underset{\sim}{n} \propto \left(\bar{\sigma}_{11} - \frac{\gamma}{\lambda_1}\bar{\sigma}_{21} + \frac{\gamma^2}{\lambda_1\lambda_2}\bar{\sigma}_{22}\right)\,, \tag{56}$$

so for $\gamma\bar{\sigma}_{21} \neq 0$, zero normal traction requires that either $\bar{\sigma}_{11}$ or $\bar{\sigma}_{22}$ is non-zero, and adopting $(54)_3$,

$$\bar{\sigma}_{11} = \frac{\gamma}{\lambda_1}\bar{\sigma}_{21}\,,\quad \bar{\sigma}_{22} = 0\,. \tag{57}$$

It is found that neither λ_1 nor λ_2 can remain constant during the above shear loading governed by the relation (52).

Corresponding to (53), noting $\underset{\sim}{D} + \underset{\sim}{W} = \dot{\underset{\sim}{F}}\,\underset{\sim}{F}^{-1}$,

$$\underset{\sim}{B} = \begin{pmatrix} \lambda_1^2+\gamma^2 & \gamma\lambda_2 \\ \gamma\lambda_2 & \lambda_2^2 \end{pmatrix},\quad \underset{\sim}{F}^{-1} = \begin{pmatrix} \lambda_1^{-1} & -\gamma(\lambda_1\lambda_2)^{-1} \\ 0 & \lambda_2^{-1} \end{pmatrix},\quad \begin{matrix} B_{33} = (\lambda_1\lambda_2)^{-2} \\ F_{33}^{-1} = \lambda_1\lambda_2 \end{matrix}\,, \tag{58}$$

$$\underset{\sim}{S} = \frac{1}{3}\begin{pmatrix} 2\lambda_1\bar{\sigma}_{11}-\lambda_2\bar{\sigma}_{22}+2\gamma\bar{\sigma}_{21} & 3\lambda_2\bar{\sigma}_{21} \\ 3\lambda_2\bar{\sigma}_{21} & 2\lambda_2\bar{\sigma}_{22}-\lambda_1\bar{\sigma}_{11}-\gamma\bar{\sigma}_{21} \end{pmatrix},\quad S_{33} = -(S_{11}+S_{22})\,, \tag{59}$$

$$\underset{\sim}{D} + \underset{\sim}{W} = \begin{pmatrix} \lambda_1^{-1}\dot{\lambda}_1 & \lambda_2^{-1}\dot{\gamma}-\gamma(\lambda_1\lambda_2)^{-1}\dot{\lambda}_1 \\ 0 & \lambda_2^{-1}\dot{\lambda}_2 \end{pmatrix},\quad (\underset{\sim}{D}+\underset{\sim}{W})_{33} = -\frac{(\lambda_1\lambda_2)^{\cdot}}{\lambda_1\lambda_2}\,, \tag{60}$$

$$K_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} + \gamma^2,\quad K_2 = \lambda_1^2\lambda_2^2 + \lambda_1^{-2} + \lambda_2^{-2} + \gamma^2\lambda_1^{-2}\lambda_2^{-2}. \tag{61}$$

For small total strain, $|e_1 = 1-\lambda_1,\ e_2 = 1-\lambda_2,\ \gamma| \ll 1$, neglecting e_1^3, e_2^3, γ^3,

$$K_1 = K_2 = 3 + 4(e_1^2 + e_2^2 + e_1e_2) + \gamma^2\,, \tag{62}$$

so that dependence on K_1 and on K_2 cannot be distinguished to this order of magnitude. Further, the shear condition (57) to lead order is

$$\bar{\sigma}_{11} = 0 \ , \quad \bar{\sigma}_{22} = 0 \ , \tag{63}$$

but $\bar{\sigma}_{11}$ contributes to the lead order diagonal components of the stress deviator (59):

$$\underset{\sim}{S} = \frac{1}{3}\begin{pmatrix} 4\gamma\bar{\sigma}_{21} & 3\bar{\sigma}_{21} \\ 3\bar{\sigma}_{21} & -2\gamma\bar{\sigma}_{21} \end{pmatrix} \ , \quad S_{33} = -\tfrac{2}{3}\gamma\bar{\sigma}_{21}, \ J_2 = \bar{\sigma}_{21}^2 \ . \tag{64}$$

The small strain approximation is now applied to each of the four loading histories shown in Table 1.

During the initial constant uni-axial stress $\bar{\sigma}_{11} = -\bar{\sigma}_o$, $\lambda_2 = \lambda_3 = \lambda_1^{-\frac{1}{2}}$, $\gamma = 0$, $e_2 = -\frac{1}{2}e_1$, $B_{11} = 1 - 2e_1$, $K_1 - 3 = 3e_1^2$, $S_{11} = -\frac{2}{3}\bar{\sigma}_o$, $J_2 = \frac{1}{3}\bar{\sigma}_o^2$, $D_{11} = -\dot{e}_1$, $\underset{\sim}{W} \equiv 0$, $tr(\underset{\sim}{S}\,\underset{\sim}{D}) = \bar{\sigma}_o\dot{e}_1$, $(\underset{\sim}{S}\,\underset{\sim}{D})_{11} = \frac{2}{3}\bar{\sigma}_o\dot{e}_1$, $\dot{S}_{11} = \frac{2}{3}\bar{\sigma}_o\dot{e}_1$, and (52) becomes

$$\dot{e}_1(3\phi_1 + 4\psi\bar{\sigma}_o) = 2\bar{\sigma}_o - 6\omega_1 e_1, \ e_1(0) = \bar{\sigma}_o/E_o << 10^{-2}, \tag{65}$$

which is the small strain approximation of (33). With the response coefficients (51), ϕ_1, ψ, and $2\bar{\sigma}_o - 6\omega_1 e_1$, each have a factor $(1-\alpha)$, so the differential equation (65) is independent of α, and integrating over $(0, t_1)$ gives the same $e_1(t)$, $e_2(t) = -\frac{1}{2}e_1(t)$ for all α. Unloading at $t = t_1$ gives a negligible elastic strain jump

$[e_1] = -\bar{\sigma}_o/E_o$, $[e_2] = \frac{1}{2}\bar{\sigma}_o/E_o$, followed by constant e_1, e_2 over (t_1, t_2). Reloading to $\bar{\sigma}_{11} = -\bar{\sigma}_o$ for $t > t_2$ recovers the elastic strain $[e_1] = \bar{\sigma}_o/E_o$, $[e_2] = -\frac{1}{2}\bar{\sigma}_o/E_o$, and continues the response as if unloading had not taken place, since the same differential equation (65) holds with initial condition $e_1(t_2+) = e_1(t_1-), = e_1^*$. Figure 6(b) shows the strain histories $e_1(t)$, $e_2(t)$ when $\bar{\sigma}_o = 10^5 Nm^{-2}$ and $t_1 = 200$ hours compared with $t_m(\bar{\sigma}_o) = 119{\cdot}4$ hours. Since the initial loading passed the minimum strain-rate time t_m, the response from the unloaded state does not show a minimum strain rate, nor, of course, any of the initial loading features. This is a crucial feature of a viscoelastic solid dependence on a reference configuration: <u>identical loading tests on samples with different pre-loading histories from the reference configuration exhibit different responses.</u>

A distinct feature is that of the <u>induced anisotropy</u>, or material asymmetry, <u>in the new unloaded configuration</u>, even when the material is isotropic in the reference configuration. Consider the alternative uni-axial stress loading $\bar{\sigma}_{22} = -\bar{\sigma}_o$ for $t > t_2$. There is an elastic strain jump $[e_2] = \bar{\sigma}_o/E_o$, $[e_1] = -\frac{1}{2}\bar{\sigma}_o/E_o$, at $t = t_2$, so that

$$e_2(t_2+) = e_2(t_1-) + \tfrac{3}{2}\bar{\sigma}_o/E_o, = -\tfrac{1}{2}e_1^* \tag{66}$$

since $\bar{\sigma}_o \ll |E_o e_2|$. In $t > t_2$,

$$\begin{aligned} e_1 - e_1^* &= -\tfrac{1}{2}(e_2 + \tfrac{1}{2}e_1^*), \quad K_1 - 3 = 3e_2^2 + \tfrac{9}{4}e_1^{*2}, \\ \dot{e}_2(3\phi_1 + 4\psi\bar{\sigma}_o) &= 2\bar{\sigma}_o - 6\omega_1 e_2 . \end{aligned} \tag{67}$$

While $(67)_3$ is the same differential equation for $e_2(t)$ in $t > t_2$ as the above reloading equation (65) for $e_1(t)$, the equations for the corresponding axial strain changes $y_2 = e_2 + \frac{1}{2}e_1^*$ and $y_1 = e_1 - e_1^*$ are not the same, and the invariant K_1-3 entering the coefficients is a different function of y_2 and y_1 respectively in the two cases.

By (65)

$$K_1-3 = 3(y_1^2 + 2y_1e_1^* + e_1^{*2}), \quad \dot{y}_1(3\phi_1 + 4\psi\bar{\sigma}_o) = 2\bar{\sigma}_o - 6\omega_1y_1 - 6\omega_1e_1^*, \tag{68}$$

with $y_1(t_2) = 0$, and by (67),

$$K_1-3 = 3(y_2^2 - y_2e_1^* + e_1^{*2}), \quad \dot{y}_2(3\phi_1 + 4\psi\bar{\sigma}_o) = 2\bar{\sigma}_o - 6\omega_1y_2 + 3\omega_1e_1^*, \tag{69}$$

with $y_2(t_2) = 0$. $K_1-3 = 3e_1^{*2}$ at $t = t_2$ in both cases but has different dependence y_1 and y_2, so the differential equations for y_1 and y_2 are not the same, even when $\omega_1 = 0$ $(\alpha = 0)$. Thus, $y_2 \neq y_1$ in $t > t_2$, and the material is exhibiting a different response to loads applied from the unloaded configuration in the OX_2 and OX_1 directions. Figure 7(b), (c) show the results for $\alpha = 0$ and $\alpha = 0{\cdot}5$, comparing $e_2(t)$ in $t > t_2$ for the $\bar{\sigma}_2$ loading to $\hat{e}_1(t) = e_1(t)$ for $\bar{\sigma}_1$ loading relative to the same starting value at t_2, that is, comparing y_2 and y_1. The difference y_2-y_1 reflects the degree of induced anisotropy. The effects of changing α, hence ω_1, are not as marked as the K_1-3 change in the arguments of ϕ_1 and ψ, because of the strongly non-linear function f. In linear viscoelasticity all coefficients are constant, but

ice exhibits strongly non-linear response even at small strains. For the model (51), $|\bar{\sigma}_o| << |\phi_1 e|$ since $\bar{\sigma}_o << |E_o e|$, and the ψ term makes negligible contribution to this small strain constant stress result. It does, however, have a significant role when $\dot{S} \neq 0$, as in the constant strain-rate response (35).

Now consider an initial constant shear load $\bar{\sigma}_{21} = \bar{\tau}_o$ applied over $(0,t_1)$. In the small strain approximation the lead order invariants and stress deviator are given by (62) and (64), and neglecting $|\gamma,e_1,e_2|$ compared to unity and noting $(\dot{e}_1,\dot{e}_2) \leqslant O(\dot{\gamma})$,

$$\underset{\sim}{B} = \begin{pmatrix} 1-2e_1 & \gamma \\ \gamma & 1-2e_2 \end{pmatrix}, \quad \underset{\sim}{D} = \begin{pmatrix} -\dot{e}_1 & \frac{1}{2}\dot{\gamma} \\ \frac{1}{2}\dot{\gamma} & -\dot{e}_2 \end{pmatrix}, \quad \underset{\sim}{W} = \begin{pmatrix} 0 & \frac{1}{2}\dot{\gamma} \\ -\frac{1}{2}\dot{\gamma} & 0 \end{pmatrix},$$

$$\dot{\underset{\sim}{S}} = \frac{1}{3}\begin{pmatrix} 4\dot{\gamma}\bar{\tau}_o & 0 \\ 0 & -2\dot{\gamma}\bar{\tau}_o \end{pmatrix}, \quad tr(\underset{\sim}{S}\,\underset{\sim}{D}) = \bar{\tau}_o\dot{\gamma} \quad . \tag{70}$$

Then the three independent components of (52) are to lead order:

$$\phi_1\dot{e}_1 + \tfrac{2}{3}\psi\bar{\tau}_o\dot{\gamma} = -(\tfrac{4}{3}\bar{\tau}_o\gamma + 2\omega_1 e_1),$$

$$\phi_1\dot{e}_2 + \tfrac{2}{3}\psi\bar{\tau}_o\dot{\gamma} = \tfrac{2}{3}\bar{\tau}_o\gamma - 2\omega_1 e_2 , \tag{71}$$

$$\phi_1\dot{\gamma} + 2\psi\bar{\tau}_o(\dot{e}_1+\dot{e}_2-\tfrac{4}{3}\gamma\dot{\gamma}) = 2(\bar{\tau}_o - \omega_1\gamma).$$

Now $|\psi\bar{\tau}_o| << |\phi_1\gamma|$ for the model (51), so the ψ term in the shear component $(71)_3$ is negligible, but if $|\dot{e}_1,\dot{e}_2| << \dot{\gamma}$, the ψ and ϕ_1 terms of $(71)_{1,2}$ may be comparable. By $(51)_1$, $(71)_3$, $|\omega_1\gamma| < O(\bar{\tau}_o)$ and

$\phi_1\dot{\gamma} = O(\bar{\tau}_o)$, so by $(51)_3$, $(71)_{1,2}$ we see that $\phi_1\dot{e}_1$, $\phi_1\dot{e}_2$ are given by terms of order $\bar{\tau}_o^2/E$, $\bar{\tau}_o\gamma$, $\bar{\tau}_o e/\gamma$, and hence $|(\dot{e}_1,\dot{e}_2)/\dot{\gamma}| \ll 1$ if $e \ll \gamma$. Initially, the jump ratios $[e_1]/[\gamma]$, $[e_2]/[\gamma]$ are negligible because of the approximations (63), which suggests that $|\dot{e}_1,\dot{e}_2| \ll \dot{\gamma}$ on $(0,t_1)$. Figure 8(b),(c) show the strain response for $\alpha = 0$ and $\alpha = 0{\cdot}5$ when $\bar{\tau}_o = 3^{-\frac{1}{2}} \times 10^5 Nm^{-2}$ (same J_2 as axial case), determined by integrating the three equations (71) simultaneously; indeed e_1 and e_2 remain negligible on $(0,t_1)$.

When the shear load is removed at $t = t_1$ there are negligible elastic strain jumps followed by constant strain on (t_1,t_2) and negligible strain jumps on reloading at $t = t_2$. But since e_1 and e_2 remain negligible for $0 \le t \le t_2+$, subsequent dependence of K_1-3 is symmetric in the axial strain changes $e_1 - e_1(t_2+)$ and $e_2 - e_2(t_2+)$, where $e_1(t_2+) = e_2(t_2+) = 0$, and also the respective axial loading equations. Hence the response $e_1(t)$ to constant stress $\bar{\sigma}_{11} = -\bar{\sigma}_o$ in $t > t_2$ is the same as $e_2(t)$ for $\bar{\sigma}_{22} = -\bar{\sigma}_o$; that is, the unloaded configuration after a shear cycle is symmetric to subsequent axial loading. Consider the response to $\bar{\sigma}_{11} = -\bar{\sigma}_o$ in $t > t_2$. From (62) with $e_2 = -\frac{1}{2}e_1$, $K_1-3 = 3e_1^2 + \gamma^2$, now depending on γ, and by (58), $B_{12} = B_{21} = \gamma$, so the axial equation (65) applies, coupled to γ through K_1-3, and the shear component of (52) gives

$$\phi_1\dot{\gamma} = -2\omega_1\gamma, \tag{72}$$

to be integrated simultaneously from starting values $e_1(t_2+) = 0$, $\gamma(t_2+)$. When [illegible] $(\alpha = 0)$, γ remains constant, but for $\alpha \neq 0$ there is significant shear strain decay during the axial reloading, reflecting the dependence on the strain tensor. Figure [illegible] shows these responses for $\alpha = 0$ and $\alpha = 0{\cdot}5$. Identical results for γ, e_2 are obtained if the alternative reloading $\bar{\sigma}_{22} = -\bar{\sigma}_0$ is applied. Note, however, if alternative shear reloadings $\bar{\sigma}_{21} = \bar{\tau}_0$ and $\bar{\sigma}_{31} = \bar{\tau}_0$ are applied, the response will be different because of the roles of the corresponding shear strain γ_{12} and γ_{13} in the invariant.

Concluding remarks

It has been shown that a differential operator relation defining an incompressible viscoelastic solid can describe the observed time response of ice to applied constant load and to applied constant displacement rate in uni-axial stress. Furthermore, both strain-rate and stress-rate terms are necessary, in addition to strain and stress terms, to reproduce both these responses. By comparison there is no strain dependence in the analogous incompressible viscoelastic fluid relation, but a strain-acceleration term is required, so that both forms of relation contain three independent response coefficients in uni-axial stress. In the solid model it is assumed that the response coefficients depend only on stress and strain (not on rates), and the above two responses determine only two combinations of the three response coefficients. Thus the general uni-axial response described by the model is not fully determined by these two responses, in contrast to the analogous overdetermined viscoelastic fluid model. Further

investigation of the construction of a solid model is required. The solid model does, however, in contrast to the fluid model, exhibit strain jumps when stress jumps are applied. We have derived the general jump relations from the differential tensor relation, and directly for uni-axial stress from the reduced uni-axial stress relation, by assuming that they are the limit of continuous changes taking place in a decreasing time interval.

Uni-axial response does not, of course, determine the tensor shape of the differential relation, nor the dependence of response coefficients on stress and strain tensors. Test responses involving two independent stress and two independent strain components are necessary to determine such shape. For illustration we adopt simplifying shape assumptions and construct a reduced tensor relation compatible with idealised uni-axial stress response. The same uni-axial response would give a different tensor relation if different shape assumptions are made. It is shown in general that the response from a pre-loaded distorted state is anisotropic, even when the initial configuration is isotropic. Such induced anisotropy is demonstrated by the responses of the reduced model to loading-unloading-reloading cycles when the reloading is applied in different directions. Both uni-axial and shear loading is considered for the initial load stage. Furthermore, reloading which repeats the initial loading shows the change of response which is caused by strain dependence in a solid model, not present in a fluid model. Correlation of test data with solid models must therefore indentify the sample configuration with respect to an undistorted state. If ice completely relaxes on

unloading, given sufficient time, then a test sample would have the original configuration on formation, isotropic or anisotropic. The model adopted for illustration exhibited no relaxation on unloading.

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Figure Captions

1. Typical response for constant load test on ice:

   a) creep curve,

   b) strain-rate v. time.

2. Typical stress-strain response for constant displacement-rate test on ice.

3. Strain relaxation after complete unloading:

   (i) no viscoelastic relaxation,

   (ii) partial viscoelastic relaxation,

   (iii) complete viscoelastic relaxation.

4. Idealised response for constant load tests:

   a) normalised creep-curve,

   b) - $f(e)$ --- $f(\hat{t})$.

5. Calculated stress-strain response for four constant displacement rates w and different $\hat{E}(e)$.

   —— $\hat{E} = E_o = 9 \times 10^9 \text{Nm}^{-2}$,

   ---- $\hat{E} = E_o e^{-160e} + E_1$,

   -·-· $\hat{E} = E_o e^{-800e} + E_1$.

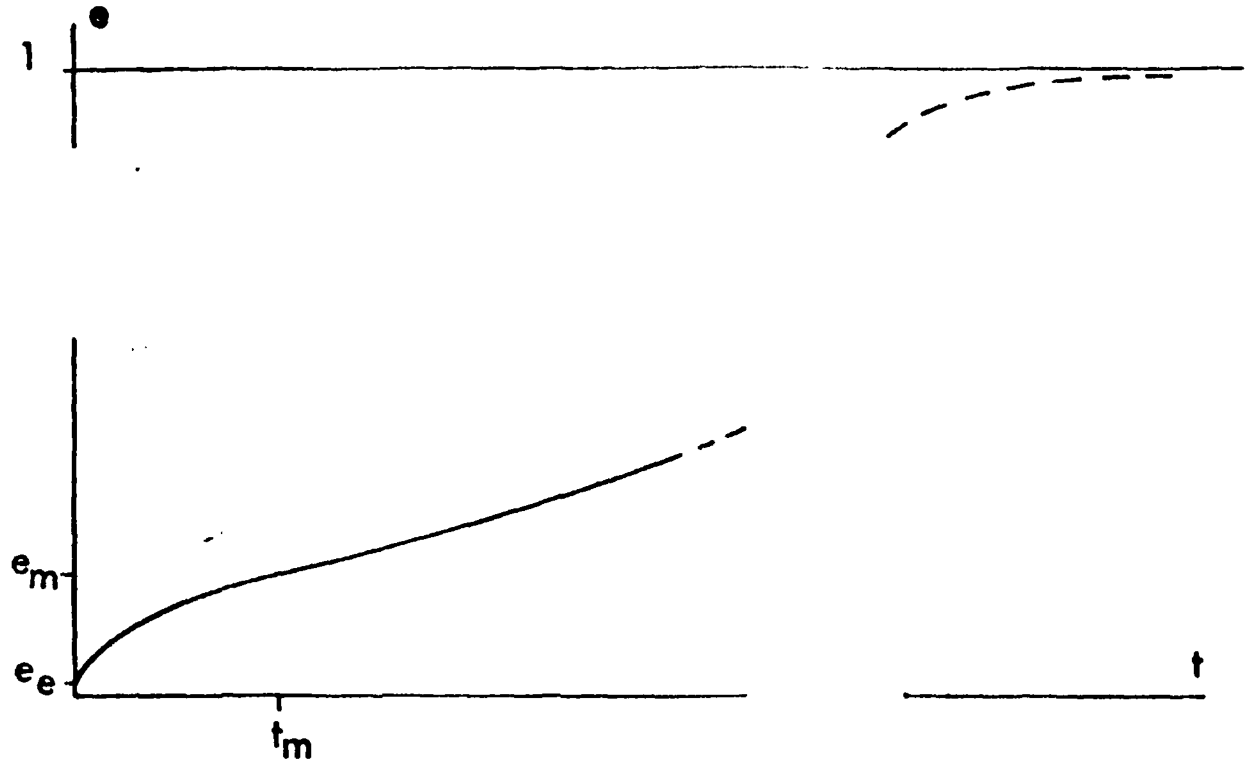
6. Response to load-unload-repeat load in uni-axial stress, all α .

7. Response to load-unload-new direction reload in uni-axial stress, for $\alpha = 0$, (b), $\alpha = 0{\cdot}5$ (c). Comparison of e_2 and $\hat{e}_1$ in $t > t_2$ reflects the induced anisotropy in states at $t = t_2$.

8. Response to shear load-unload followed by uni-axial load for $\alpha = 0$. (b), $\alpha = 0{\cdot}5$ (c).

Figure 1. Typical response for constant load test on ice

(a) creep curve



(b) strain-rate v. time.

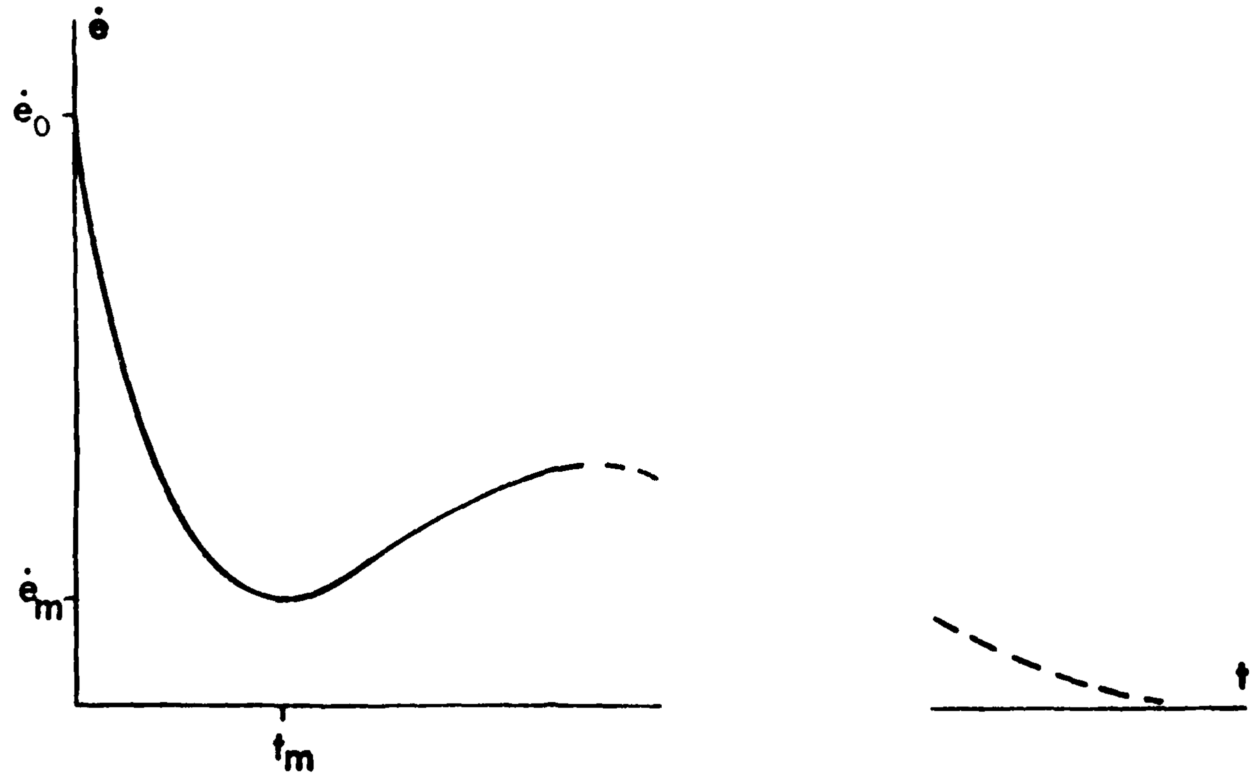


Figure 2. Typical stress-strain response for constant displacement-rate test on ice.

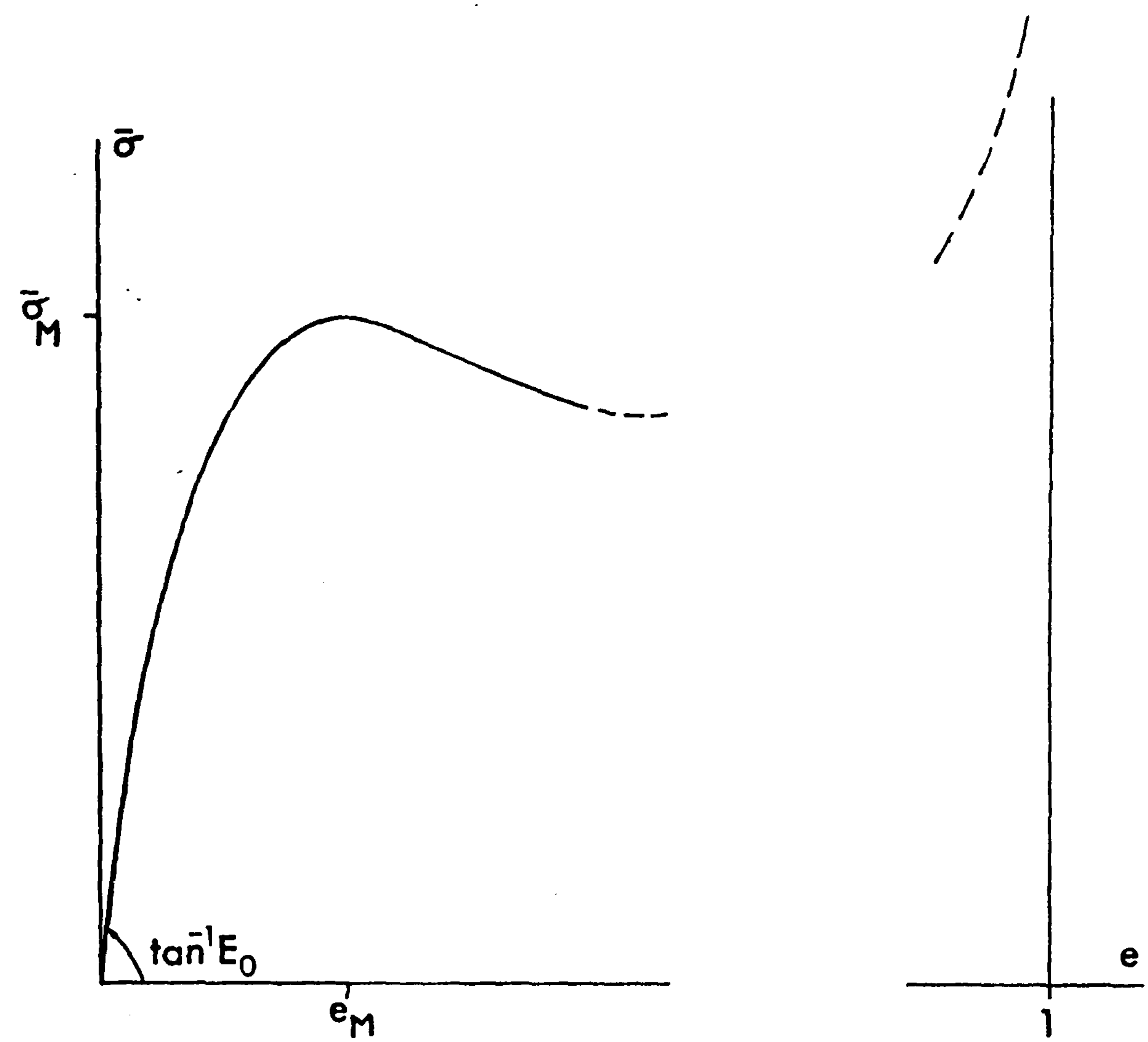


Figure 3: Strain relaxation after complete unloading:
(i) no viscoelastic relaxation,
(ii) partial viscoelastic relaxation,
(iii) complete viscoelastic relaxation.

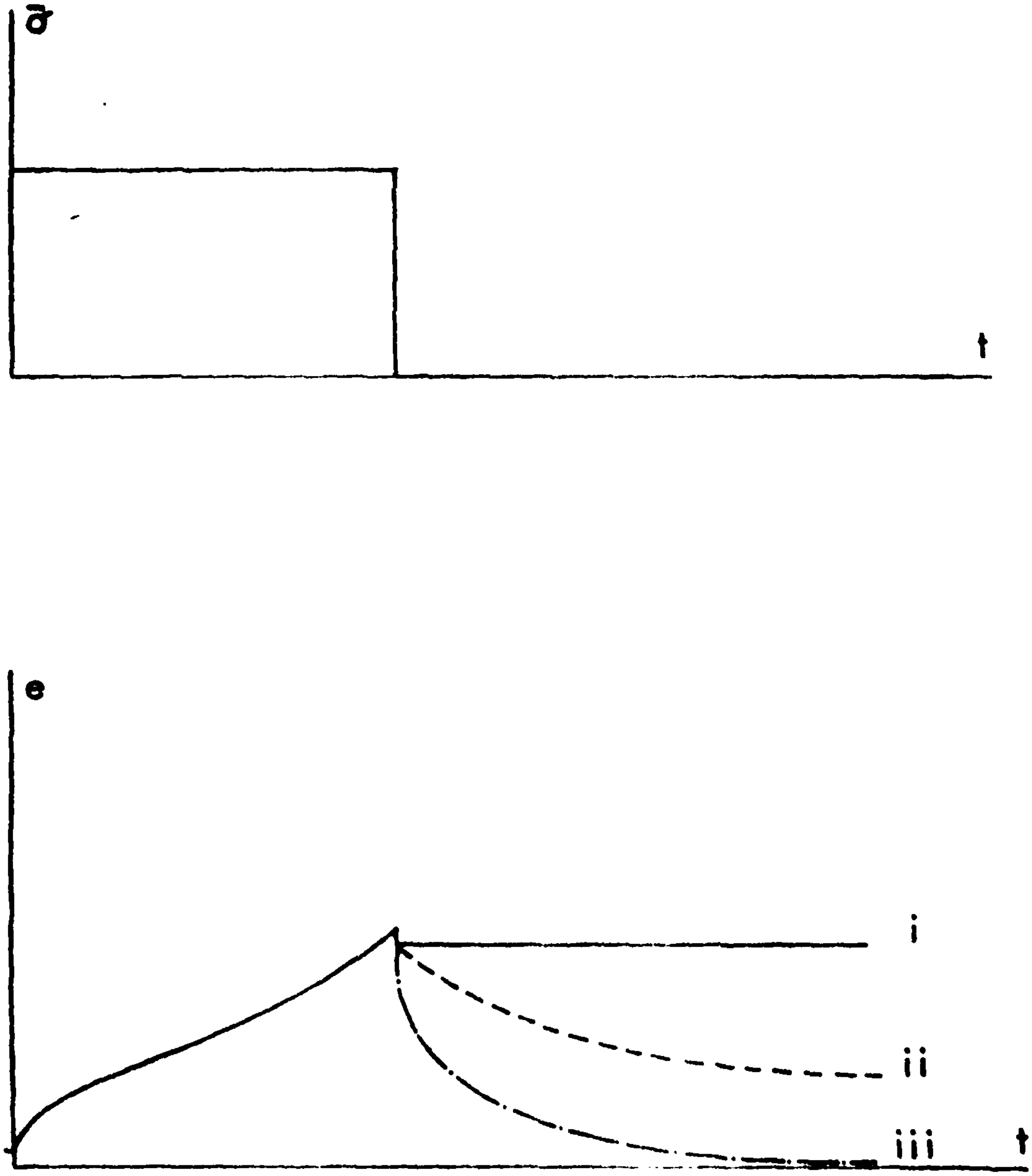
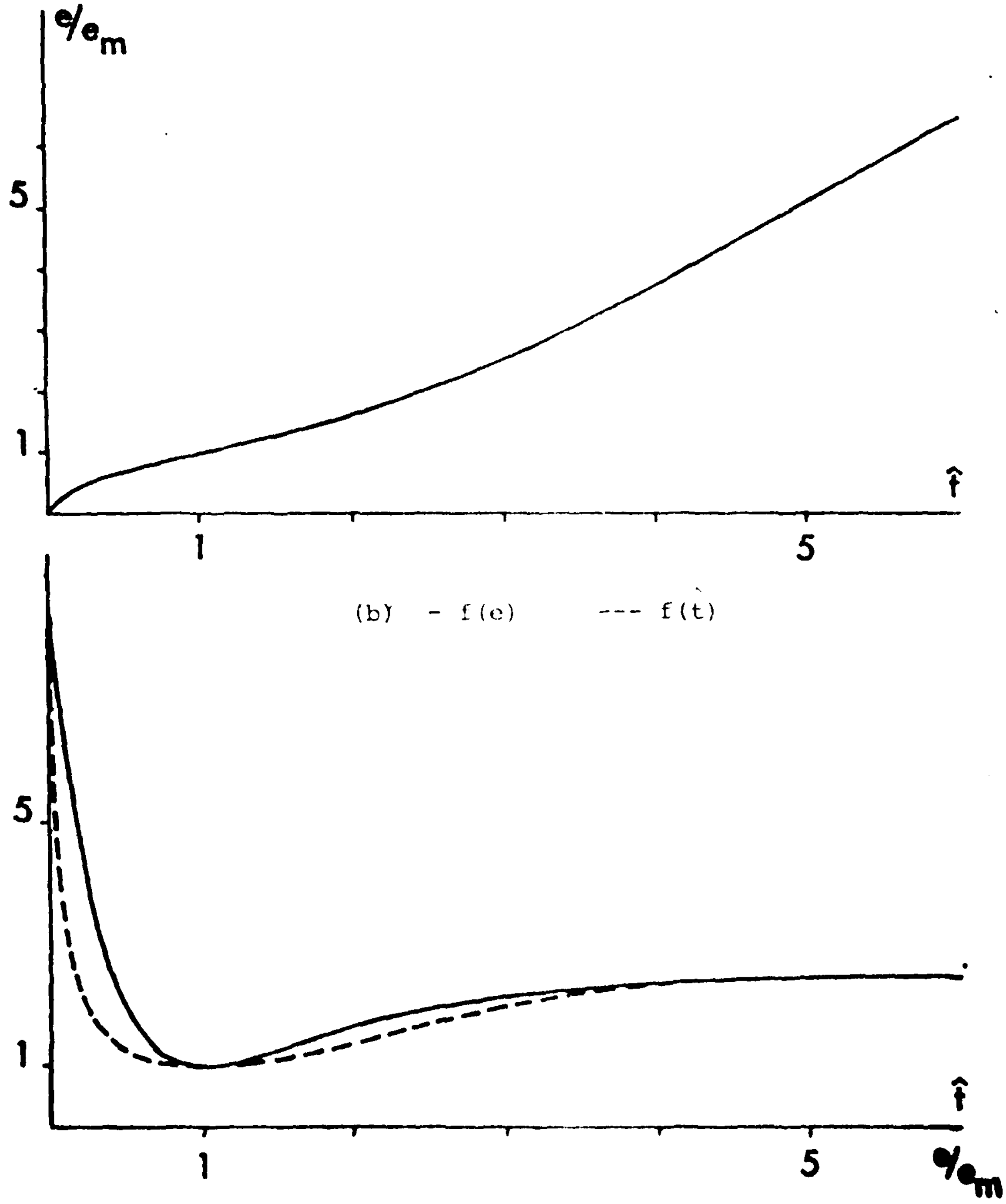


Figure 4. Idealised response for constant load tests:

(a) normalised creep curve



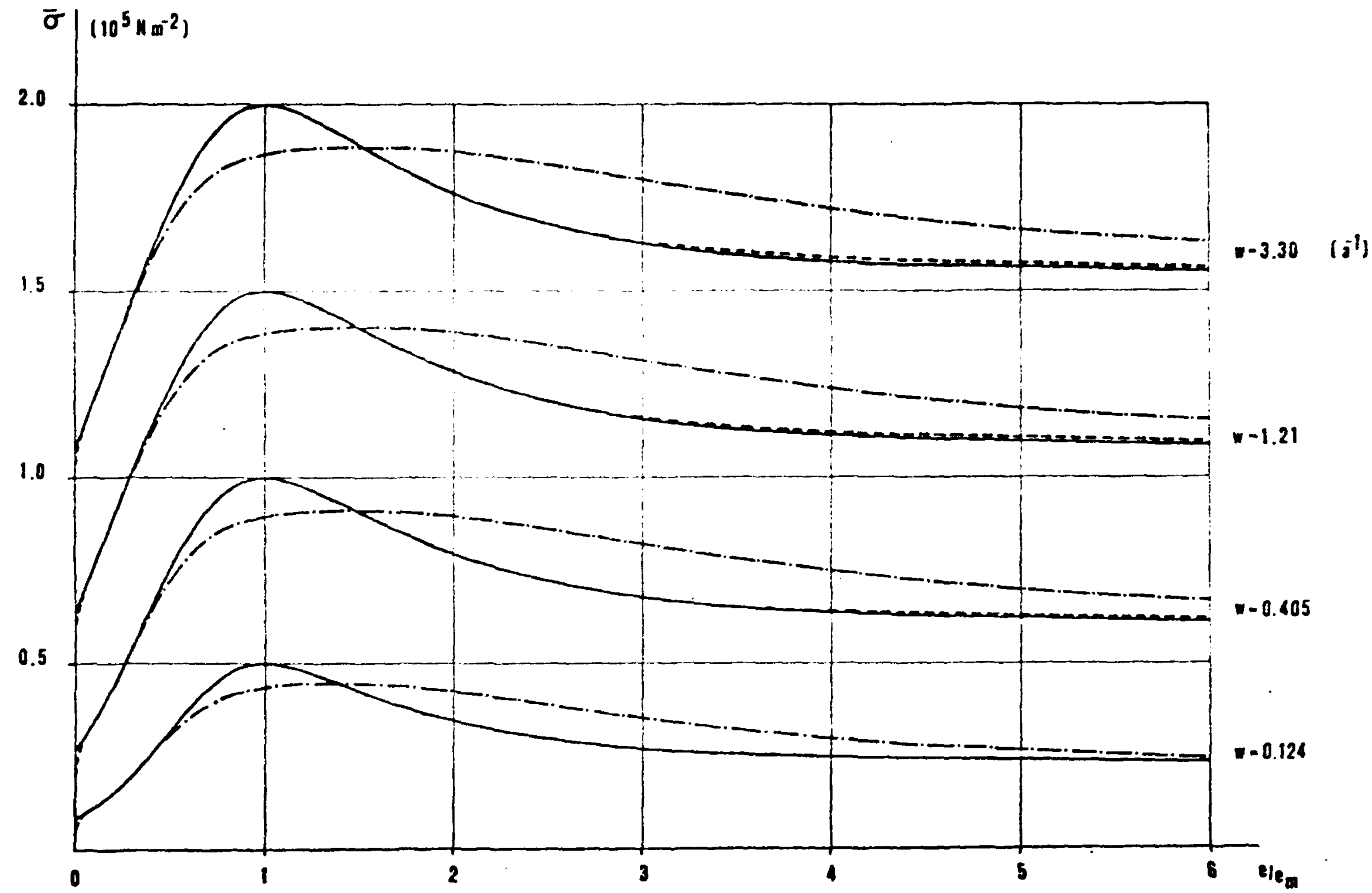
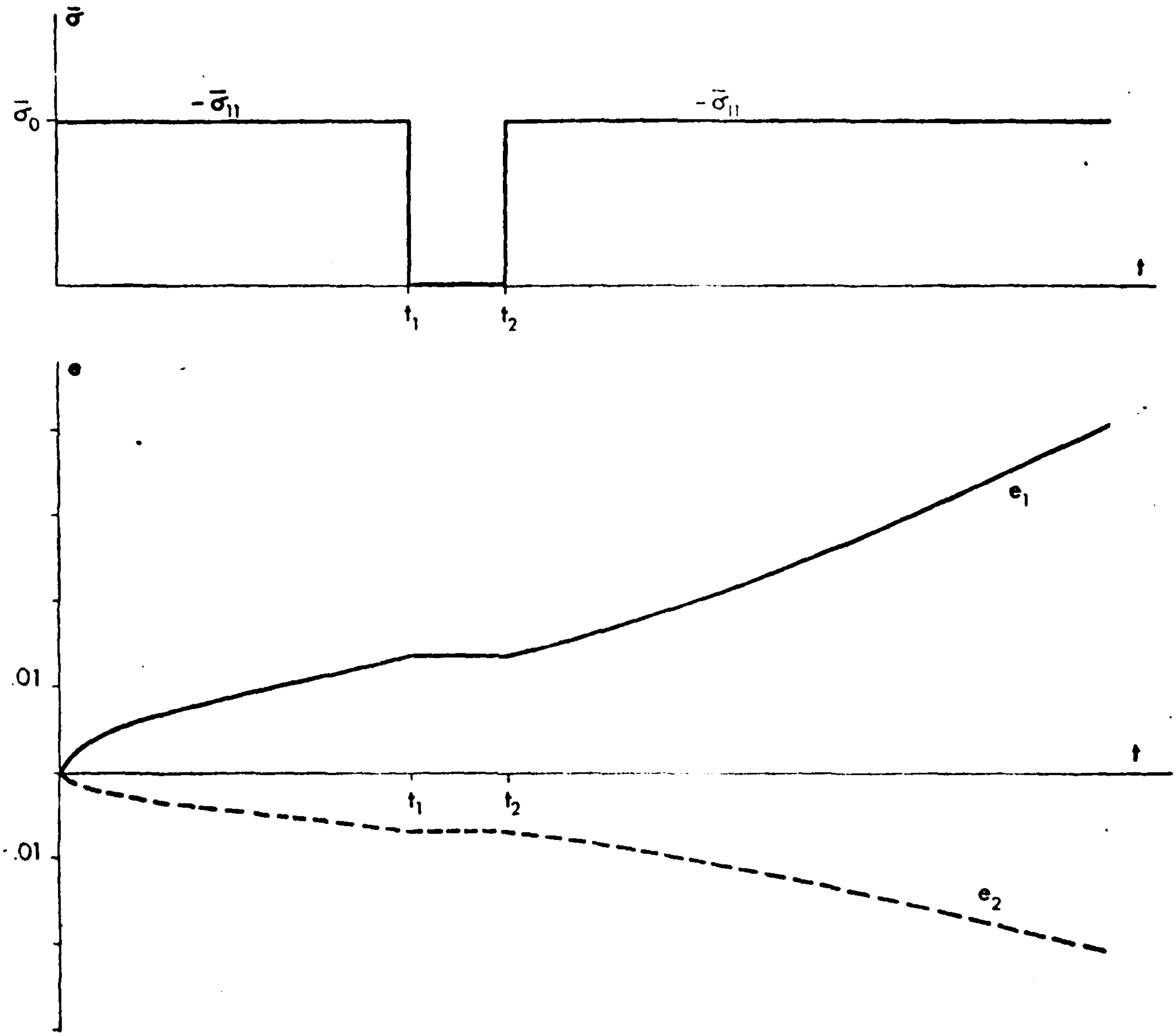


Figure 5. Calculated stress-strain response for four constant displacement rates w and different $\hat{E}(e)$:

——— $\hat{E} = E_o = 9 \times 10^9 Nm^{-2}$, ---- $\hat{E}' = E_o e^{-160e} + E_1$,

-·-·- $\hat{E} = E_o e^{-800e} + E_1$.

Figure 6. Response to load-unload-repeat load in uni-axial stress, all α.



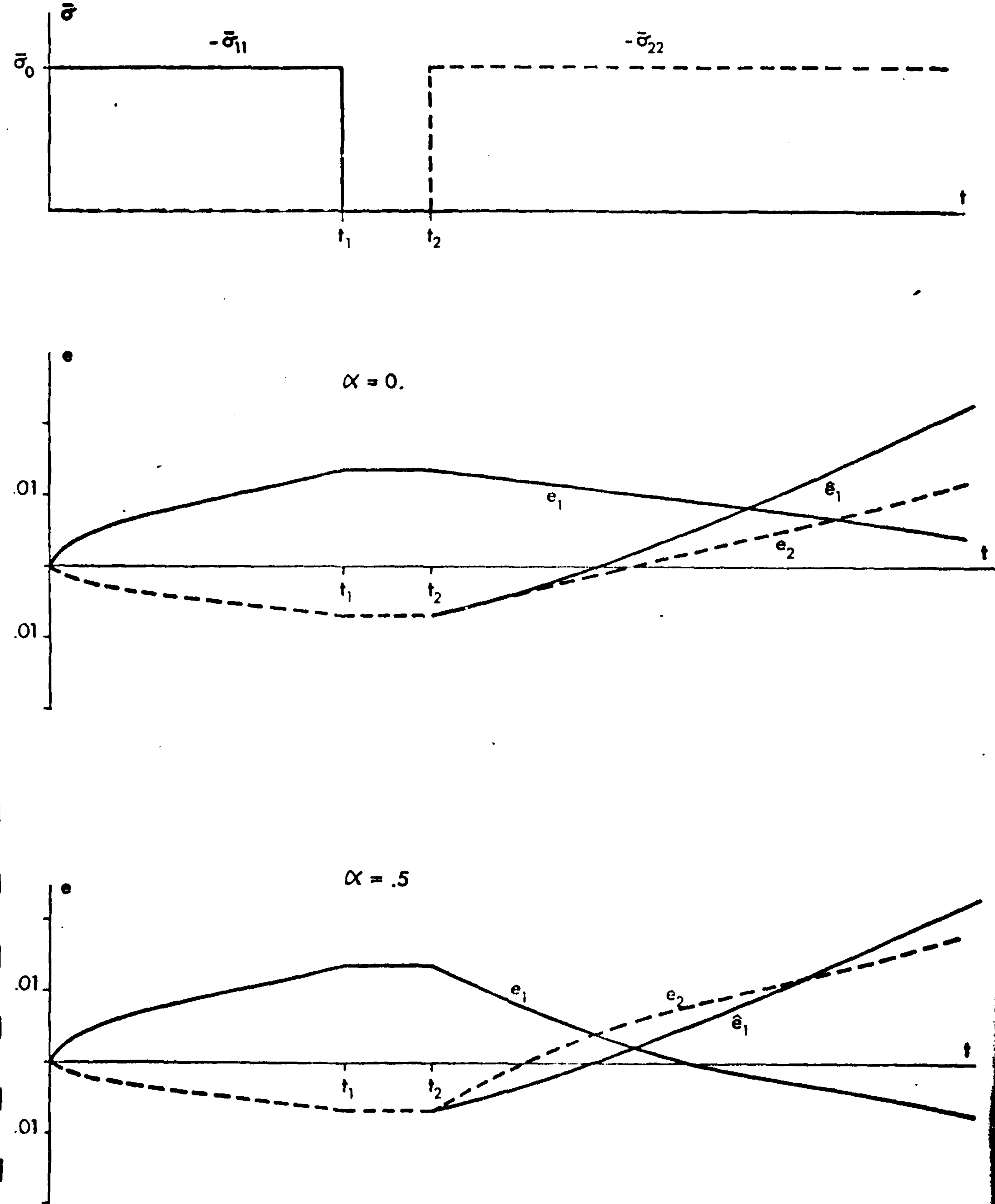


Figure 7. Response to load-unload-new direction reload in uni-axial stress, for $\alpha = 0$, (b), $\alpha = 0{\cdot}5$ (c). Comparison of e_2 and $\hat{e}_1$ in $t > t_2$ reflects the induced anisotropy in state at $t = t_2$.

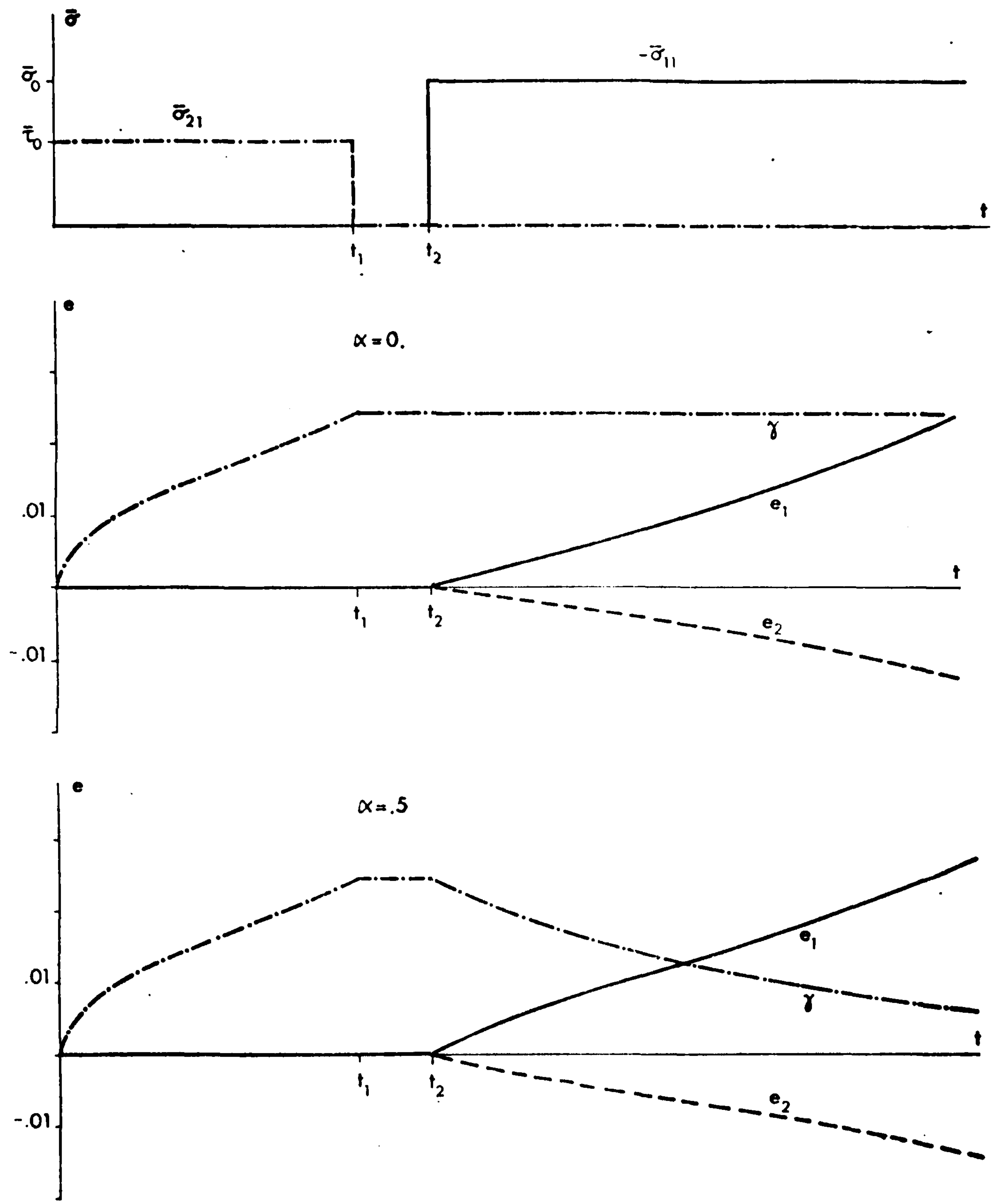


Figure 8. Response to shear load-unload followed by uni-axial load for $\alpha = 0.$ (b), $\alpha = 0.5$ (c).

PART II

Single Integral Representations For Non-Linear Viscoelastic Solids

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Abstract

The finite linear viscoelastic solid and single integral representation with non-linear dependence on history are investigated in uni-axial stress. Both integral kernels in the stress formulation are determined by single-step constant strain tests, and both kernels in the strain formulation are determined by single-step constant stress tests. Single integral stress and strain formulations are not equivalent. The stress histories required to maintain constant strain-rate for both models are determined from the Volterra integral equations given by the strain formulations once their kernels are determined by constant stress tests. However, known constant strain-rate response does not determine the kernels. Examples are presented to show that variation of the kernel within a given qualitative shape can lead to different shapes of constant strain-rate response, so that both constant stress and constant strain-rate tests may be necessary to deduce the optimum single integral approximation, in preference to multi-step stress tests. It is shown that the apparently simpler finite linear viscoelastic model requires a far lengthier numerical algorithm to solve the Volterra equation, and leads to non-unique and physically unacceptable response, in comparison with the more flexible non-linear history dependence which yields unique acceptable responses.

1. Introduction

A general frame indifferent representation of a viscoelastic material is

$$\hat{\underset{\sim}{\sigma}}(t) \equiv \underset{\sim}{R}^T(t)\,\underset{\sim}{\sigma}(t)\,\underset{\sim}{R}(t) = \underset{\sim}{\mathcal{F}}[\underset{\sim}{E}(\overset{t}{\underset{-\infty}{\tau}})], \tag{1.1}$$

where $\underset{\sim}{\sigma}$ is the Cauchy stress, $\underset{\sim}{E}$ is a strain defined by $\frac{1}{2}(\underset{\sim}{F}^T\underset{\sim}{F} - \underset{\sim}{1})$ where $\underset{\sim}{F}$ is the deformation gradient from a given reference configuration, and $\underset{\sim}{R}$ is the rotation from the reference configuration {see, for example, Dill (1975)}. That is, the rotated stress $\hat{\underset{\sim}{\sigma}}$ at time t depends on the history of strain $\underset{\sim}{E}(\tau)$, $-\infty < \tau \le t$, which can be termed a stress formulation. Alternatively, if (1.1) can be inverted, there is a strain formulation

$$\underset{\sim}{E}(t) = \underset{\sim}{\mathcal{H}}[\hat{\underset{\sim}{\sigma}}(\overset{t}{\underset{-\infty}{\tau}})] \tag{1.2}$$

in which the strain at time t depends on the history of the rotated stress. Dill (1975) notes that the inverse exists for linear viscoelastic materials, but not for linearly viscous fluids. It is supposed to exist for general viscoelastic materials. Any material symmetries of the reference configuration impose restrictions on the tensor functionals $\underset{\sim}{\mathcal{F}}$ and $\underset{\sim}{\mathcal{H}}$.

For an incompressible material

$$\det(\underset{\sim}{1} + 2\underset{\sim}{E}) \equiv \det(\underset{\sim}{F}^T\underset{\sim}{F}) = 1, \tag{1.3}$$

and

$$\hat{\underset{\sim}{\sigma}}' \equiv \underset{\sim}{R}^T\underset{\sim}{S}\,\underset{\sim}{R} = \underset{\sim}{\mathcal{F}}'[\underset{\sim}{E}(\overset{t}{\underset{-\infty}{\tau}})], \quad \underset{\sim}{E}(t) = \underset{\sim}{\mathcal{H}}'[\hat{\underset{\sim}{\sigma}}'(\overset{t}{\underset{-\infty}{\tau}})], \tag{1.4}$$

where the stress deviator $\underset{\sim}{S}$ is defined by

$$\underset{\sim}{S} = \underset{\sim}{\sigma} + p\underset{\sim}{1}, \quad p = -\frac{1}{3}\mathrm{tr}\,\underset{\sim}{\sigma}, \tag{1.5}$$

and the mean pressure p is a workless constraint not determined by the deformation. Lockett (1972) suggests that a given pure shear strain history (of an incompressible material) may require isotropic pressures p, in contrast to the shear relations (1.4) based on the workless constrain postulate. In this case, the incompressibility condition (1.3) should be adopted as a good kinematic approximation (small dilation compared to typical shear strains), and the full stress relation (1.1) or (1.2) applied.

Theoretical modelling of non-linear viscoelastic behaviour has focussed mainly on multiple integral expansions of the stress functional $\underset{\sim}{\mathcal{F}}$, (1.1), with kernels depending only on time lapse $t - \tau$ (Green and Rivlin 1957, 1960; Green, Rivlin and Spencer 1959, Pipkin 1964), and specific representations for isotropic response have been formulated. Such models are usually truncated after the triple integral, but still require a large programme of tests to determine the kernel functions. Lockett (1965, 1972) has presented theoretical programmes of multiple step constant stress creep tests sufficient to determine the kernels of the third order one-dimensional and three-dimensional models, with corresponding constant strain tests for a third order expansion of the strain functional $\underset{\sim}{\mathcal{H}}$ (1.2). The third order expansion of the strain formulation $(1.4)_2$ for the

incompressible case is analysed further by Lockett and Stafford (1969). Motivation for the truncated stress integral expansion is that the current strain and weighted integration of past strain (fading memory) are small, Dill (1975), though the expansion is an exact theory which may describe the large strain response of some material. Truncation of the strain expansion has less motivation, and truncations of both expansions to the same order are not equivalent formulations, and require different numbers of kernels in the incompressible case (Lockett 1972).

An alternative expansion in a small strain difference history from the current strain, $\underset{\sim}{E}(\tau) - \underset{\sim}{E}(t)$, leads to integral series with kernels depending on the fixed strain point $\underset{\sim}{E}(t)$, (Dill 1975). The first term, linear in the strain difference history, defines a finite linear viscoelastic material (Coleman and Noll 1961), given by

$$\hat{\underset{\sim}{\sigma}}(t) = \underset{\sim}{K}[\underset{\sim}{E}(t), 0]\,\underset{\sim}{E}(t) + \int_{-\infty}^{t} \dot{\underset{\sim}{K}}[\underset{\sim}{E}(t),\, t-\tau]\underset{\sim}{E}(\tau)d\tau\,, \tag{1.6}$$

where $\underset{\sim}{K}(\underset{\sim}{E},t)$ is a fourth order tensor function, zero on $t < 0$, and $\dot{\underset{\sim}{K}}$ denotes the derivative of $\underset{\sim}{K}(\underset{\sim}{E},t)$ with respect to its second argument, supposed continuous in both arguments. The first term of a similar strain expansion in stress difference history gives

$$\underset{\sim}{E}(t) = \underset{\sim}{C}[\hat{\underset{\sim}{\sigma}}(t), 0]\hat{\underset{\sim}{\sigma}}(t) + \int_{-\infty}^{t} \dot{\underset{\sim}{C}}[\hat{\underset{\sim}{\sigma}}(t), t-\tau]\hat{\underset{\sim}{\sigma}}(\tau)\, d\tau\,, \tag{1.7}$$

where $\underset{\sim}{C}[\hat{\underset{\sim}{\sigma}},t]$ is a fourth order tensor, zero on $t < 0$, and $\dot{\underset{\sim}{C}}$ denotes derivative with respect to the second argument. It is convenient to describe (1.7) also as a finite linear viscoelastic material.

The linear viscoelastic relations for infinitesimal strain $\underset{\sim}{E}$ and infinitesimal rotation $\underset{\sim}{R} - \underset{\sim}{1}$ are given by (1.6) and (1.7) with $\hat{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma}$ and

$$\underset{\sim}{K}(\underset{\sim}{E},t) \equiv \underset{\sim}{K}(t) , \quad \underset{\sim}{C}(\hat{\underset{\sim}{\sigma}},t) \equiv \underset{\sim}{C}(t) . \tag{1.8}$$

In this case (1.6) and (1.7) are equivalent with the relaxation function $\underset{\sim}{K}(t)$ and creep function $\underset{\sim}{C}(t)$ related by the linear Volterra integral equation

$$\underset{\sim}{K}(0)\ \underset{\sim}{C}(t) + \int_0^\infty \underset{\sim}{K}'(t-\tau)\ \underset{\sim}{C}(\tau)d\tau = \underset{\sim}{1}\ H(t) \tag{1.9}$$

where $\underset{\sim}{1}$ is the unit fourth order tensor and $H(t)$ is the Heaviside step function. $\underset{\sim}{K}(t)\underset{\sim}{E}_o$ is the stress response to constant strain $\underset{\sim}{E}_o H(t)$, and $\underset{\sim}{C}(t)\underset{\sim}{\sigma}_o$ is the strain response to constant stress $\underset{\sim}{\sigma}_o H(t)$.

From the finite relations (1.6) and (1.7), $\underset{\sim}{K}(\underset{\sim}{E}_o,t)\underset{\sim}{E}_o$ is the stress response to constant strain $\underset{\sim}{E}_o H(t)$, and $\underset{\sim}{C}(\hat{\underset{\sim}{\sigma}}_o,t)\hat{\underset{\sim}{\sigma}}_o$ is the strain response to constant stress $\hat{\underset{\sim}{\sigma}}_o H(t)$, but there is no integral equation analogous to (1.9) which determines a function of two variables $\underset{\sim}{C}(\hat{\underset{\sim}{\sigma}},t)$ given $\underset{\sim}{K}(\underset{\sim}{E},t)$, or vice-versa. Let $\underset{\sim}{K}(\underset{\sim}{E}_o,t)\underset{\sim}{E}_o = \overline{\underset{\sim}{K}}(\underset{\sim}{E}_o,t)$ be given, then (1.7) gives

$$\underset{\sim}{E}_o = \underset{\sim}{C}[\overline{\underset{\sim}{K}}(\underset{\sim}{E}_o,t),0]\overline{\underset{\sim}{K}}(\underset{\sim}{E}_o,t) + \int_0^t \dot{\underset{\sim}{C}}[\overline{\underset{\sim}{K}}(\underset{\sim}{E}_o,t),t-\tau]\overline{\underset{\sim}{K}}(\underset{\sim}{E}_o,\tau)d\tau \tag{1.10}$$

which, for each fixed $\underset{\sim}{E}_o$, is an integral equation to determine

$\underset{\sim}{C}[\underset{\sim}{\bar{K}}(\underset{\sim}{E}_o, \bar{t}), t]$ for $0 \leq t < \bar{t}$, $\bar{t} > 0$. As different $\underset{\sim}{E}_o$ are chosen, the stress argument $\underset{\sim}{\bar{K}}(\underset{\sim}{E}_o, \bar{t})$ will be repeated at different values $\bar{t}$, but the corresponding solutions of (1.10) at the same time $t < \bar{t}_{min}$ will not, in general, be identical; that is, there is no $\underset{\sim}{C}(\hat{\underset{\sim}{\sigma}}, t)$ which satisfies (1.10) for all $\underset{\sim}{E}_o$. This becomes more evident when a numerical algorithm to solve the one-dimensional form of (1.10) is attempted. Hence, unlike the fully linear theory (1.8), (1.6) and (1.7) are not generally equivalent formulations. Given the kernel $\underset{\sim}{C}$, of course, (1.7) is an integral equation for the stress history $\hat{\underset{\sim}{\sigma}}(t)$ necessary to support any prescribed strain history $\underset{\sim}{E}(t)$, and a one-dimensional numerical algorithm is presented later. Corresponding results follow from the stress formulation (1.6).

Both forms of multiple integral expansion discussed above are motivated by a strain history or difference history remaining small, with corresponding strain expansions less firmly based. Pipkin and Rogers (1968) point out that strong non-linearity may then require a high-order multiple integral expansion, with a prohibitive experimental programme, and, furthermore, very difficult stress and deformation analysis. They propose that the first single integral should account, if possible, for the strong non-linearity by allowing non-linear dependence of the kernel on the strain or stress history, with successive multiple integral terms viewed as refinements. Both stress and strain expansions are constructed so that the single integral describes exactly the response to single-step constant strain or constant

stress tests, and the multiple integral terms vanish identically for such histories. The successive multiple integral terms are constructed so that the nth order kernels are determined by an n-step constant strain or constant stress test with higher-order terms vanishing identically. Thus, extra terms can be added to refine the correlation with extra multi-step response without disturbing the integral series already constructed, in strong contrast to the earlier multiple integral expansions in which all the kernels change as the level of truncation changes. The single integral representations are

$$\hat{\underset{\sim}{\sigma}}(t) = \underset{\sim}{G}[\underset{\sim}{E}(t),0] + \int_{-\infty}^{t} \dot{\underset{\sim}{G}}[\underset{\sim}{E}(\tau),\ t-\tau]d\tau \quad , \tag{1.11}$$

$$\underset{\sim}{E}(t) = \underset{\sim}{J}[\hat{\underset{\sim}{\sigma}}(t),0] + \int_{-\infty}^{t} \dot{\underset{\sim}{J}}[\hat{\underset{\sim}{\sigma}}(\tau),\ t-\tau]d\tau \quad , \tag{1.12}$$

where the relaxation function $\underset{\sim}{G}[\underset{\sim}{E},t]$ and creep function $\underset{\sim}{J}[\hat{\underset{\sim}{\sigma}},t]$ are second order tensors depending on the tensors $\underset{\sim}{E}$ and $\hat{\underset{\sim}{\sigma}}$ respectively, as well as on time. $\underset{\sim}{G}(\underset{\sim}{E},t)$ and $\underset{\sim}{J}(\hat{\underset{\sim}{\sigma}},t)$ are defined to be zero for $\underset{\sim}{E} \equiv \underset{\sim}{0}$ and $\hat{\underset{\sim}{\sigma}} \equiv \underset{\sim}{0}$ respectively, then $\underset{\sim}{G}(E_o,t)$ is the stress response to constant strain $\underset{\sim}{E}_o H(t)$, and $\underset{\sim}{J}[\hat{\underset{\sim}{\sigma}}_o,t]$ is the strain response to constant stress $\hat{\underset{\sim}{\sigma}}_o H(t)$. An integral equation corresponding to (1.10) is given for the constant strain response, and as before the kernel $\underset{\sim}{G}$ does not determine the kernel $\underset{\sim}{J}$, nor does $\underset{\sim}{J}$ determine $\underset{\sim}{G}$, so (1.11) and (1.12) are not equivalent. The creep-relaxation relations constructed by Findley, Lai, and Onaran (1976) are based on iterative

approximations for $\underset{\sim}{G}$ and $\underset{\sim}{J}$ quadratics in the histories.

Lockett (1972) argues that successive multiple integral terms in the Pipkin and Rogers (1968) scheme are not readily determined, since the number of arguments in the kernels increases rapidly because of the dependence on both time and strain or stress history, but that this representation is a useful basis for simplifications such as kernels seperable in time and strain or stress history. Pipkin and Rogers (1968) have used single-step constant uni-axial stress data for a PVC plastic to determine the one-dimensional form of the single integral representation (1.2), which is then shown to predict good agreement for a five-step stress history test. Single integral representations for polypropylene and polyethylene also predict good agreement for two-step stress histories, but for a polyurethane foam do not agree well with a four-step stress test. They also argue that multi-step stress tests provide a more severe trial of the single integral representation than smooth stress history response. Clearly single integral representations are the most tractable for stress and deformation analysis and require the least extensive experimental programme, and the forms (1.11) and (1.12) which incorporate non-linear history dependence offer most flexibility, with some good response correlations already demonstrated by Pipkin and Rogers (1968).

We now investigate the strain representation (1.12) and the corresponding finite linear representation (1.7) in uni-axial stress. Given the scalar kernels J and C defining the same single-step constant stress response, the respective stress histories required to maintain constant strain-rate are determined by Volterra integral equations. Distinct numerical algorithms

are necessary. It is found that the apparently simpler finite linear viscoelastic relation requires a considerably lengthier and less satisfactory algorithm than the general non-linear relation, which follows also for other loadings. Furthermore, in examples with C determined by a smooth constant stress response, the constant strain-rate stress history for the finite linear viscoelastic relation becomes non-unique and all branches become physically unacceptable. In contrast, the general non-linear relation yields a unique physically acceptable solution, and accuracy and stability of the numerical algorithm is tested by application to the linear viscoelastic relations for which exact solutions can be derived. However, variation of the kerenel J within a given qualitative shape can lead to widely differing shapes of stress history for constant strain-rate. The same effect is also exhibited by solutions of the linear viscoelastic relations when similar kernel variation is investigated, and is not peculiar to the chosen non-linear kernel. Thus, when constant strain-rate response is known as well as constant stress response, the "best" single kernel is not necessarily the exact single step constant stress response, as in the Pipkin and Rogers (1968) scheme, but rather that required to approximate both responses. That is, the trial of a single integral representation should be multi-type tests rather than multi-step tests of the same type. The constant strain-rate response is a smooth stress history, which, in our examples, does provide significant information. Our inverse methods do not show, however, how the constant strain-rate response can be used directly to "optimise" the kernel J, nor to construct

double-integral terms to refine the model, in contrast to the Pipkin and Rogers (1968) scheme, but have revealed the possible failure of models constructed by single-type tests.

2. Uni-axial stress

Consider a uni-axial stress configuration with one non-zero stress component σ inducing an axial strain $E_{11} = e$, equal lateral strains $E_{22} = E_{33}$, zero shear strains E_{ij} $(i \neq j)$, and no rotation $(\underset{\sim}{R} = \underset{\sim}{1})$, where the components refer to rectangular Cartesian axes Ox_i $(i = 1,2,3)$. For an incompressible material E_{22} is determined by (1.3):

$$E_{22} = \tfrac{1}{2}\{(1+2e)^{-\frac{1}{2}} - 1\} , \qquad (2.1)$$

but for a compressible material must be determined by $\sigma_{22} = \sigma_{33} = 0$ in the constitutive equation (1.1) or (1.3). The axial relations corresponding to (1.6) and (1.7) for the finite linear viscoelastic solid are

$$\sigma(t) = K[e(t),0]e(t) + \int_{-\infty}^{t} \dot{K}[e(t),t-\tau]e(\tau)d\tau , \qquad (2.2)$$

$$e(t) = C[\sigma(t),0]\sigma(t) + \int_{-\infty}^{t} \dot{C}[\sigma(t),t-\tau]\sigma(\tau)d\tau , \qquad (2.3)$$

and the linear viscoelastic relations are given by $K = K(t)$, $C = C(t)$, with

$$K(0)C(t) + \int_{0}^{\infty} K'(t-\tau)C(\tau)d\tau = C(0)K(t) + \int_{0}^{\infty} C'(t-\tau)K(\tau)d\tau = H(t) . \quad (2.$$

The non-linear viscoelastic relations (1.11) and (1.12) become

$$\sigma(t) = G[e(t),0] + \int_{-\infty}^{t} \dot{G}[e(\tau),t-\tau]d\tau \quad , \tag{2.5}$$

$$e(t) = J[\sigma(t),0] + \int_{-\infty}^{t} \dot{J}[\sigma(\tau),t-\tau]d\tau \quad . \tag{2.6}$$

The kernels $K(t)$, $C(t)$ in the linear viscoelastic theory, depending only on time, are determined by the response to any applied stress or strain history. However, dependence of K on $e(t)$ in (2.2) and of G on $e(\tau)$ in (2.5), implies that K and G can be determined uniquely only by families of constant strain tests, $e(t) \equiv e_o H(t)$ with a sequence of e_o to cover the required strain range. The stress response is then precisely $K[e_o,t]e_o$ and $G(e_o,t)$ respectively, described as the relaxation function. Similarly, the creep function,or strain response to constant stress $\sigma(t) \equiv \sigma_o H(t)$, is $C[\sigma_o,t]\sigma_o$ and $J[\sigma_o,t]$ from (2.3) and (2.6) respectively. Other smooth strain or stress histories, including constant strain-rate $e(t) = rtH(t)$ or constant stress-rate $\sigma(t) = stH(t)$, do not lead to unique kernels unless, of course, the particular single integral representation is an exact description of the class of smooth responses chosen. Furthermore, there is no simple correlati of the function $K[e(t),t-\tau]$, $\tau \le t$, nor of the function $G[e(\tau),t-\tau]$, $\tau \le t$, with the stress response $\sigma(t)$ for a general history $e(t)$, and corresponding conclusions hold for the strain formulations (2.3) and (2.6). Also, the non-uniform stress

response $\sigma(t)$ to constant strain does not determine the kernel C in (2.3) nor the kernel J in (2.6), and the non-uniform strain response $e(t)$ to constant stress does not determine K in (2.2) nor G in (2.5). For multiple integral terms, only multi-step constant strain or multi-step constant stress tests allow direct correlation with the kernels of a stress or strain formulation respectively, and the two formulations cannot be related.

Now suppose that the kernels C and J have been determined by families of constant stress tests, so that the representations (2.3) and (2.6) describe the same constant stress response. The Volterra integral equations (2.3) and (2.6) for the respective stress histories $\sigma(t)$ corresponding to a prescribed strain history $e(t)$ can be solved numerically to investigate the different responses predicted by the two representations. We will present the algorithms and illustrations for constant strain-rate $e(t) = rt$ as an example of a smooth loading history used in practice. Given K and G from constant strain tests, the Volterra integral equations (2.2) and (2.5) similarly determine the responses to an applied stress history $\sigma(t)$.

First consider the non-linear relation (2.6) with $e(t) = rtH(t)$, where r is a constant, so that $\sigma(t) = 0$ for $t < 0$. Define

$$t_i = i\delta \, , \quad \sigma_i = \sigma(t_i), \qquad i = 0,1,2,\ldots, \tag{2.7}$$

where δ is a small time interval, and interpret $t_o = 0+$. By (2.6)

$$0 = J[\sigma_o, 0] \quad . \tag{2.8}$$

If $J[\sigma,0] > 0$ for all $\sigma > 0$; that is, there is a positive strain jump at $t = 0$ when a positive stress jump is applied which is the typical response of a solid; then

$$\sigma_o = 0 \tag{2.9}$$

and the stress history starts smoothly. If, however, the initial strain jump is negligible compared to typical creep strains, so $J[\sigma,0] \equiv 0$ is adopted as an approximation, then σ_o is not determined by (2.8), and may be non-zero. Applying the trapezoidal rule over the first time interval, (2.6) gives

$$r\delta = J[\sigma_1, 0] + \tfrac{1}{2}\delta\{\dot{J}[\sigma_o, \delta] + \dot{J}[\sigma_1, 0]\} \quad . \tag{2.10}$$

Now if $J[\sigma,0] \equiv 0$, $\dot{J}[\sigma,0] > 0$, the limit $\delta \to 0$ gives

$$\dot{J}[\sigma_o, 0] = r \, . \tag{2.11}$$

We assume that the initial creep rate $\dot{J}[\sigma,0]$ at constant stress σ increases with σ, so (2.11) yields a unique solution σ_o for each r. Now σ_o is determined by (2.9) or (2.11), and in both cases (2.10) is an implicit equation for σ_1. Continued application of the trapezoidal rule over successive time intervals gives for $n = 2,3,\ldots$

$$nr\delta = J[\sigma_n,0] + \frac{1}{2}\delta\,\dot{J}[\sigma_n,0]$$

$$+ \delta\{\frac{1}{2}\dot{J}[\sigma_o,t_n] + \sum_{i=1}^{n-1}\dot{J}[\sigma_i,t_{n-i}]\} \quad , \tag{2.12}$$

which is an implicit equation for σ_n once σ_i $(i = 0,1,\ldots n-1)$ are determined by previous steps, and σ_n occurs in only two terms.

The algorithm (2.10), (2.12), with (2.9) or (2.11), can be applied to the linear viscoelastic relations by setting $J[\sigma(\tau),t] \equiv J(t)\sigma(\tau)$ in (2.6) so that $J[\sigma_i,t_j]$ becomes $J(t_j)\sigma_i$. An exact solution of (2.6) can then be derived (by Laplace transforms) for exponential type creep functions, and examples have shown the time interval magnitude, relative to the time scale of the creep function, necessary for the algorithm to yield an accurate solution. Further, with the assumption that $\dot{J}[\sigma,0]$ increases with σ, and the assumption $J[\sigma,0]$ increases with σ, describing expected response, (2.10), (2.12) yield unique solutions $\sigma_1,\sigma_2,\ldots$.

For the finite linear viscoelastic relation (2.3), constant strain-rate given by $e(t) = rtH(t)$ yields the initial condition

$$0 = C[\sigma_o,0]\sigma_o \, , \tag{2.13}$$

with solution $\sigma_o = 0$ if $C[\sigma,0] > 0$ for all σ, and the first step relation

$$r\delta = C[\sigma_1,0]\sigma_1 + \tfrac{1}{2}\delta\{\dot{C}[\sigma_1,\delta]\sigma_o + \dot{C}[\sigma_1,0]\sigma_1\}. \tag{2.14}$$

When $C[\sigma,0] \equiv 0$, $\dot{C}[\sigma,0] > 0$, the limit $\delta \to 0$ gives

$$\dot{C}[\sigma_o,0]\sigma_o = r \quad . \tag{2.15}$$

As before we assume $\dot{C}[\sigma,0]$ increases with σ, then (2.15) yields a unique solution σ_o for each r. Again, (2.14) is an implicit equation for σ_1, but here all three terms in C have argument σ_1, and in particular the term $\dot{C}[\sigma_1,\delta]$ involves the kernel evaluated at the unknown σ_1 and time $\delta \neq 0$. Over successive time intervals the algorithm gives for $n = 2,3,\ldots$

$$\begin{aligned} nr\delta &= C[\sigma_n,0]\sigma_n + \tfrac{1}{2}\delta\{\dot{C}[\sigma_n,t_n]\sigma_o + \dot{C}[\sigma_n,0]\sigma_n\} \\ &\quad + \delta\sum_{i=1}^{n-1}\dot{C}[\sigma_n,t_{n-i}]\sigma_i \end{aligned} \tag{2.16}$$

which is an implicit equation for σ_n once σ_i $(i = 0,1,\ldots n-1)$ are determined by previous steps. However, every term in C involves the unknown argument σ_n and n terms involve $\dot{C}[\sigma_n,t_{n-i}]$ evaluated at $t_1,t_2,\ldots t_n \neq 0$. Thus, solution of the implicit equation (2.16) at each step becomes increasingly lengthy as n increases, in contrast to (2.12) where σ_n occurs in only two terms, and those evaluated only at $t = 0$. Properties of $C[\sigma,0]$ no longer allow any conclusion about each solution σ_n since evaluations at σ_n and each time step occur. The collapse of (2.16) for the linear viscoelastic relation is too dramatic to provide a sensible test of the numerical algorithm, but solutions have been repeated with different time intervals to check consistency.

Our illustrations show that non-uniqueness can indeed occur for smooth, physically sensible, creep functions C, so not only is (2.16) unattractive for numerical methods, but the finite linear viscoelastic parent relation (2.3) can imply non-sensible response. This situation must also arise for less simple loadings.

3. Illustrations

Constant stress and constant strain-rate in uni-axial compressive stress are the common tests used to determine the mechanical properties of ice at constant temperature, reviewed by Mellor (1980). Consider σ and e to be positive in compression. While detailed results over the large temperature range of interest (230K $\rightarrow$ 273K) have not been determined, the main features for strains up to about 0·05 are known, and typical responses are shown in Fig. 1. At constant stress σ, Fig. 1a, there is an initial small elastic strain jump $e_e = \sigma/E_0$, where E_0 is the Young's modulus at zero stress, of order $10^{10}\,\mathrm{Nm}^{-2}$, followed by a primary decelerating creep ($\ddot{e} < 0$) until time $t_m(\sigma)$ when $\ddot{e} = 0$, $\dot{e} = r_m(\sigma)$, $e = e_m(\sigma)$, then a tertiary accelerating creep ($\ddot{e} > 0$). The minimum strain-rate $r_m(\sigma)$, often referred to as secondary or steady state creep, is significantly non-linear in σ, and highly temperature dependent. A close least squares fit to laboratory data at 273.13K over the stress range $0 \le \sigma \le 9 \times 10^5\,\mathrm{Nm}^{-2}$ derived by Smith and Morland (1981) is

$$r_m(\sigma) = \sigma(0{\cdot}3336 + 0{\cdot}320\sigma^2 + 0{\cdot}0296\sigma^4) \qquad (3.1)$$

where σ is measured in units $10^5 Nm^{-2}$ and strain-rate in units a^{-1}. The strain at minimum strain-rate, $e_m(\sigma)$, is approximately 0·01, independent of σ over this stress range, and much larger than the elastic strain $e_e(\sigma)$. Figure 1b shows the corresponding strain-rate at constant stress. The typical response to constant strain-rate $\dot{e} = r$ is shown in Fig. 1c, starting smoothly at zero stress with σ increasing to a maximum $\sigma_M(r)$ at time $t_M(r)$, then decreasing. The strain at $t_M(r)$, $e_M(r) = r\,t_M(r)$, is also approximately 0·01 for a moderate range of r. While the creep at constant stress, Fig. 1a, is monotonic, the decelerating and accelerating stages shown more clearly in Fig. 1b are not common in other rheological models. There is a corresponding non-monotonic stress response at constant strain-rate, Fig. 1c, but we find that this does not follow from (2.6) for all creep functions with the above properties.

An idealised creep model for the respresentation (2.6) which reflects the qualitative features of ice deformation, adopted by Morland and Spring (1981) and Spring and Morland (1982) to illustrate differential operator relations, is given by

$$\dot{J}[\sigma,t] = r_m(\sigma)\{A - Be^{-bt/t_m} + De^{-dt/t_m}\}\ , \tag{3.2}$$

$$J[\sigma,t] = \int_0^t \dot{J}[\sigma,t]dt + \sigma/E_0\ , \tag{3.3}$$

with $r_m(\sigma)$ defined by (3.1) and $E_0 = 9 \times 10^9 Nm^{-2}$. Then making the approximation $e_m(\sigma) \equiv 0{\cdot}01$, (3.2) and (3.3) give

$$t_m(\sigma) = \frac{0{\cdot}01 - \sigma/E_o}{a\, r_m(\sigma)}\,, \quad a = A + \frac{B}{b}(e^{-b} - 1) - \frac{D}{d}(e^{-d} - 1)\,, \tag{3.4}$$

where the constant a must be positive. Also

$$J[\sigma,0] = \sigma/E_o > 0\,, \quad \dot{J}[\sigma,0] = r_m(\sigma)\{A - B + D\}\,, \tag{3.5}$$

so $J[\sigma,0]$ and $\dot{J}[\sigma,0]$ increase with σ provided that $A - B + D > 0$.

The algorithm (2.12) has been used to determine the stress history for three constant strain-rates $r = 0{\cdot}5a^{-1}$, $1a^{-1}$, $2a^{-1}$, and the three parameter sets for (3.2) and (3.4) listed in Table 1.

	I	II	III
A	2·5	2·5	2·5
B	3·6664	1·5660	1·5164
D	10·7206	15·1685	29·2772
b	0·6706	0·0380	0·0099
d	5b	200b	1000b

Table 1. Parameters for idealised creep functions

These all give the same ratio $t_I/t_M = 1{\cdot}7$, where t_I is the time to the inflexion point $(\ddot{J} = 0)$ in the tertiary creep, and satisfy $a > 0$ and $J[\sigma,0]$, $\dot{J}[\sigma,0]$ increasing in σ. From (3.2) - (3.4),

$$J(\sigma,t) = \frac{\sigma}{E_o} + \frac{0{\cdot}01 - \sigma/E_o}{a}\{A\frac{t}{t_m} + \frac{B}{b}(e^{-bt/t_m} - 1) - \frac{D}{d}(e^{-dt/t_m} - 1)\} \tag{3.6}$$

is a function of $t/t_m(\sigma)$ alone when σ/E_o is neglected. Figure 2 shows e and $\dot{e}/r_m(\sigma)$ as functions of $t/t_m(\sigma)$ for the parameter sets I, II, III; each has the required qualitative shape.

The distinction between sets I, II, and III, is seen in the stress histories they predict at constant strain-rate, shown in Fig. 3, as functions of $e = rt$. With set I there are large repeated oscillations, with set II there is a small stress rise after relaxation before the steady response, and with set III the required stable shape corresponding to Fig. 1c is obtained. If a linear viscoelastic relation with related creep function

$$J(t) = \frac{1}{E_O} + j\{At + \frac{Bt_m}{b}(e^{-bt/t_m} - 1) - \frac{Dt_m}{d}(e^{-dt/t_m} - 1)\}, \qquad (3.7)$$

where j and t_m are constants, is solved exactly for the constant strain-rate response, then oscillations during the relaxation phase arise if

$$\lambda = \frac{[A(b+d) - Bd + Db]^2}{4Ad(A - B + D)} < 1, \qquad (3.8)$$

when the elastic response $1/E_O$ is neglected. For the parameter sets I, II, III, $\lambda = 0{\cdot}11$, $1{\cdot}3$, and $3{\cdot}4$ respectively, so that small oscillations still occur during relaxation in the non-linear model at $\lambda = 1{\cdot}3$, but not at $\lambda = 3{\cdot}4$. The criterion (3.8) provides a rough guide to the likelihood of relaxation oscillations. These illustrations show, however, that creep functions of the correct qualitative shape can lead to distinct shapes of constant strain-rate response from the representation (2.6), so that an optimum approximation for the kernel J must take account of constant strain-rate response. The above procedure, of course, provides only an inverse approach, and a direct method has yet to be constructed.

The same constant stress response (3.2) - (3.4) is described by the finite linear viscoelastic relation (2.3) with

$$C[\sigma,t] \equiv J[\sigma,t]/\sigma, \quad \dot{C}[\sigma,t] \equiv \dot{J}[\sigma,t]/\sigma \ , \tag{3.9}$$

which are bounded as $\sigma \to 0$. Application of the lengthy algorithm (2.16) with the parameter set II and $r = 1a^{-1}$ leads to a non-unique stress long before the peak stress given by the relation (2.6) is reached, and the time at which this is observed numerically depends on the time interval δ. Table 2 provides an outline of the numerical results obtained with time intervals δ = 1 hour, 2 hours, 10 hours, corresponding to strain increments $1{\cdot}142 \times 10^{-4}$, $2{\cdot}284 \times 10^{-4}$, $1{\cdot}142 \times 10^{-3}$ respectively at $r = 1a^{-1}$, and the results from the non-linear relation algorithm (2.12) with δ = 2 hours for comparison. Once non-uniqueness occurs, there are three roots of the implicit equation, and each branch yields three roots at the next step, so no physically sensible solution can be deduced. The Table shows only the continuation of the minimum and maximum root branches of the first non-unique solution. Until non-uniqueness occurs the solutions for different δ are consistent, and close to the general non-linear solution as expected for such a short initial time range. This dramatic non-uniqueness property of the finite linear viscoelastic integral equation must extend to other prescribed strain histories, and the representation (2.3) would appear to have no general validity. The same situation must arise in the usual stress formulation (2.2) with some smooth kernels $K[e,t]$ when the integral equation for constant stress-rate response is solved.

t hours	$\sigma[10^5 Nm^{-2}]$, δ = 1 hour			δ = 2 hours		δ = 10 hours		δ = 2 hours (2.12)
0	0·181			0·181		0·181		0·181
1	0·183							
2	0·186			0·185				
3	0·188							
4	0·191			0·191				0·191
5	0·194							
6	0·197			0·197				
7	0·200							
8	0·203			0·203				0·202
9	0·207							
10	0·210			0·210		0·206		
11	0·214							
12	- 0·018	0·218	0·054	0·218				0·213
13	0·068	0·222						
14	- 0·200	0·226		0·266				
15	0·808	0·254						
16	- 0·351	- 0·093		- 0·021	0·236			0·226
17	1·127	1·183						
18	- 0·630	- 0·627		0·876	0·241			
19	- 0·037	- 0·033						
20	1·511	1·496		- 0·364	0·393	0·251		0·236
22	↑	↑		1·111	- 0·169			
24	σ_{min}	σ_{max}		- 0·505	1·194			0·252
26				- 0·063	- 0·493	σ_{min}	σ_{max}	
28				1·381	- 0·092	↓	↓	0·267
30				- 0·415	0·012	- 0·069	0·322	
40				↑	↑	0·903	0·442	0·316
50				σ_{min}	σ_{max}	- 0·196	0·153	

Table 2. Algorithm (2.16) solution for different time intervals δ .

4. Concluding remarks

The finite linear viscoelastic representation (2.3) or (2.2) appears to have no validity for significantly non-linear response with the features described above, namely, a monotonic strain response to constant stress with an inflexion point, or analogous stress response to constant strain. The general non-linear single integral representation (2.6), and presumably (2.5), yields integral equations with stable numerical algorithms and unique solutions in the examples treated. However, the distinct shapes of constant strain-rate response which can arise for kernels of the same shape determined by constant stress response suggests that multi-type tests, rather than multi-step tests of the same type, are necessary to obtain a valid single integral approximation. A direct procedure for correlating the kernel with multi-type response has yet to be formulated, and may prove a major hurdle, but the inverse approach used in the above examples would offer a trial and refinement method.

Acknowledgement

This research has been supported by the United States Army European Research Office in collaboration with the Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

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Figure Captions

1. Typical response of ice in uni-axial compressive stress:

   (a) strain at constant stress,

   (b) strain-rate at constant stress,

   (c) stress at constant strain-rate.

2. Strain and strain-rate at constant stress for the three models I (———), II (----), III (-·-·-).

3. Stress at constant strain-rates $r = .5a^{-1}, 1a^{-1}, 2a^{-1}$, for the models I, II, and III.

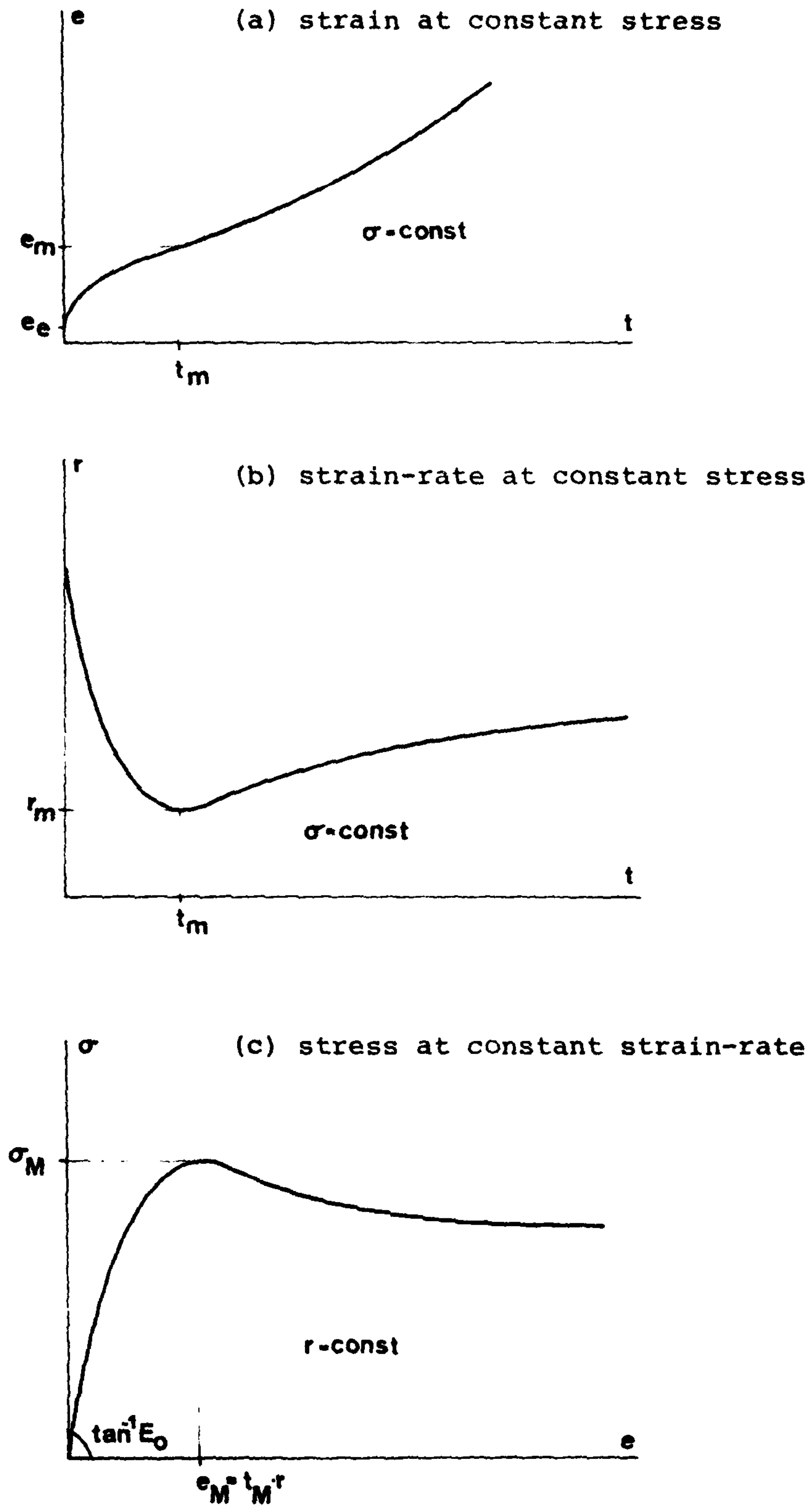

Figure 1. Typical response of ice in uni-axial compressive stress:
(a) strain at constant stress,
(b) strain-rate at constant stress,
(c) stress at constant strain-rate.

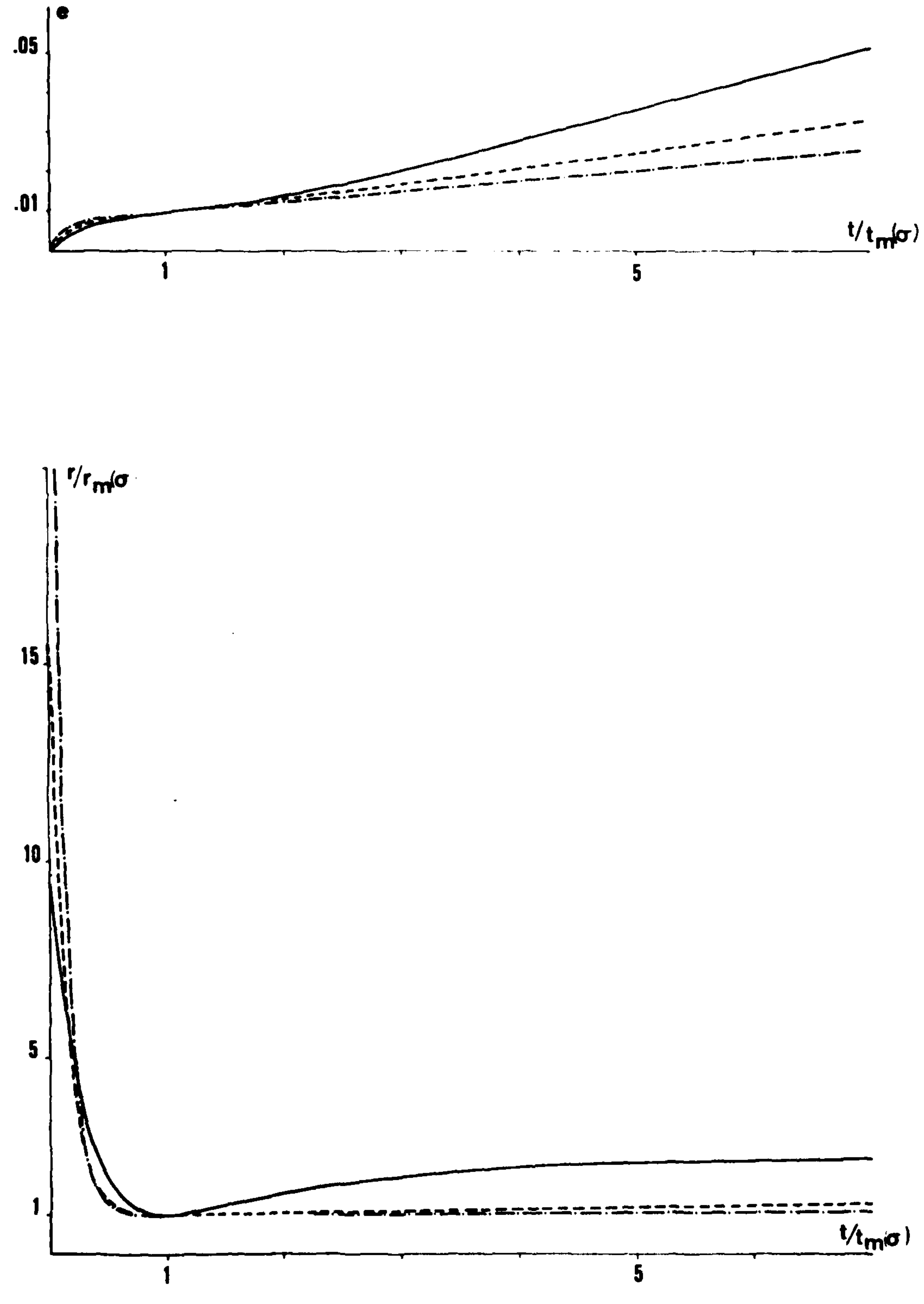

Figure 2. Strain and strain-rate at constant stress for the three models I(——), II(----), III)-·-·-).

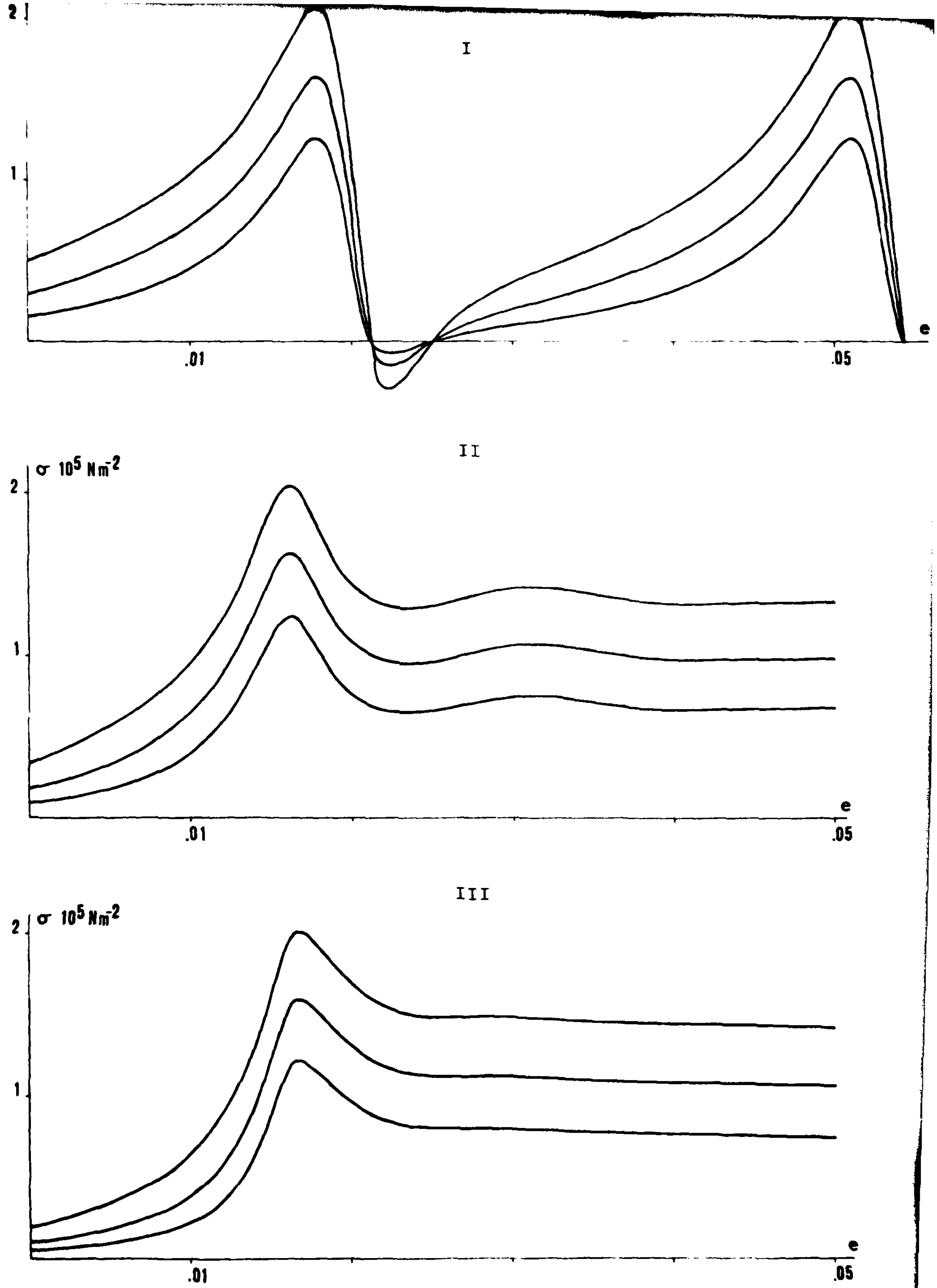

Figure 3. Stress at constant strain-rates $r = \cdot 5a^{-1}$, $1a^{-1}$, $2a^{-1}$, for the models I, II, and III.